Stationary Analysis and Optimality Conditions for
(σ, S) Policies in Multi-Commodity Inventory
Control Problems

Technical Report No. 62

SUBMITTED BY:

Nabil S. Faour

Nabil S. Faour
Research Associate

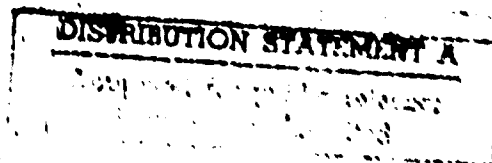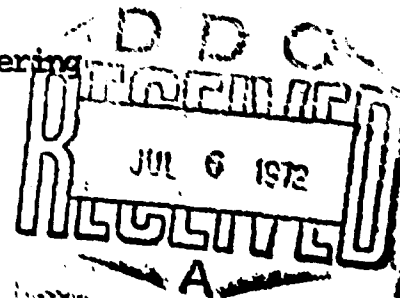*B. D. Sivazlian*

B. D. Sivazlian
Principal Investigator

APPROVED BY:

B. D. Sivazlian

B. D. Sivazlian
Editor, Project THEMIS

Department of Industrial and Systems Engineering
The University of Florida
Gainesville, Florida 32601

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An m-commodity (m ≥ 1) inventory control model with periodic review is formulated and studied when a dyadic stationary ordering (σ, S) policy in E^m is used. It is assumed that: (1) the ordering decision at the beginning of each of a sequence of periods of time is affected by a single set up cost, a linear variable ordering cost and holding and shortage costs; (2) the demands for the items in each period are identically and independently distributed continuous random variables; (3) the delivery of orders is immediate; and (4) complete backlogging of unfilled demands is allowed. Next the model is analyzed by minimizing the expression for the stationary total expected cost per period. The set of simultaneous equations used to determine the optimal policy parameters are restated in terms of a real valued function in E^m. Finally, the optimization analysis is restricted to the special case of a two-commodity inventory control problem where the demands for the items obey the exponential distribution, and the holding and shortage costs are linear. The integral equation which is used in solving for the optimal policy parameters is converted into a linear hyperbolic partial differential equation of the second order with boundary conditions. This boundary value problem is solved analytically. The solution is then used to determine the optimal policy parameters. In a numerical example a computer program is developed to determine the optimal policy parameters.

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is formulated and studied when a dyadic stationary ordering (σ, S) policy
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total expected cost per period. The set of simultaneous equations used to
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parameters.

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CHAPTER I

INTRODUCTION

1.1 Introduction

The replenishment and control of inventories is a common
practice in the fields of business, economics and management in general.
Simply, an inventory is a stock of physical goods which are held or
stored for future use, sale or production. Inventory problems are
integrated in nature. They may involve production, scheduling, distri-
bution of commodities, or combination of all. In the past twenty years,
much work has been done to formulate and apply the proper mathematical
tools for determining economic decision rules in managing the various
inventory systems. Viewed as decision problems, the analysis of such
systems essentially proceeds in three steps: (1) the formulation of a
model expressing a set of relations between a set of decision variables
whose values are to be determined; (2) the establishment of an objective
function of the decision variables so as to minimize or maximize
an economic objective; and (3) a numerical method to determine the values
of the decision variables that optimize the objective function.

In this introduction we shall restrict ourselves to models in
which decisions are made at the beginning of each of a number of equal
periods of time, and in which the demands are independently and identi-
cally distributed random variables from period to period. The decision
at the beginning of each period is affected by a fixed set up cost, a

variable ordering cost and a holding and shortage costs. Backlogging
and immediate delivery of orders are allowed.

Two principal distinct approaches have been used in analyzing
such inventory models both in theory and practice. In the first approach,
the system is viewed as a multistage decision process and the technique
of dynamic programming is employed in finding the optimal policy that
minimizes the total expected cost over the duration of the process. To
ensure that the optimal policy is of the "simple type" certain restric-
tions are imposed on the cost factors that affect the decision rules.
More precisely, when the ordering cost is linear and when the sum of
holding and shortage costs $L(x)$ is convex, the optimal policy is of the
"simple type." However, when the duration of the process is infinite.
a second approach is often used: A "simple type" ordering policy is
chosen to be used in each period. Under the chosen simple policy the
sequence of inventory levels at the beginning of each period forms a
Markov process, whose stationary behavior can be analyzed. Here, instead
of optimizing the total cost over the horizon period, which is meaningless,
we minimize the stationary total expected cost per period. The optimal
values of the decision variables are called the steady-state or sta-
tionary solutions to the inventory problem. Some of the advantages of
the stationary approach over the dynamic programming approach are:

1. The stationary approach provides us with information
   about the dependency of the optimal policies on the many
   parameters involved in the model and about sensitivity of
   costs as function of the policies, while this information is
   not provided by following the dynamic programming approach.

2. As the number of commodities increases, the number of the
   state variables increases drastically, and the use of
   dynamic programming techniques in obtaining optimal policies
   for the inventory problem will be time-consuming even when
   using the fastest computers.

3. Frequently, for low-cost high turnover items, optimal
   policies are not really required. It is sufficient to
   obtain simple analytical approximations to optimal policies,
   and these are obtained from the stationary approach.

The main concern of this research will be to find the stationary
solutions to the m-commodity (m ≥ 1) inventory problem. For the one
commodity, the theory of inventory has studied the stationary behavior
of the system and provided optimal decision rules for inventory replen-
ishment under a linear ordering cost and a convex holding and shortage
costs function. The results of the one-commodity inventory system are
valid for multi-commodity inventory system where each commodity is
treated independently.

However, in practice this independency assumption is uncommon
and it is more realistic to incorporate the interdependency in the set
up order cost of the commodities which arises in the nature of the
following types of problems.

1. Ordering different commodities at the same time from a
   common vendor may reduce the procurement ordering costs,
   thus incurring one ordering cost.

2. Several warehouses used as storage and distribution points
   are supplied by a single factory. Whenever an order to
   replenish a given commodity stored by the warehouses is

made to the factory, a fixed set up cost is incurred

whether the warehouses order individually or simultaneously.

The configuration of this system is shown in Fig. (1.1).

The two problems are similar in structure and it is practical

to take advantage of the reduced procurement cost in controlling the

inventory of the commodities. Furthermore, if one can solve the multi-

commodity problem, then the multi-warehouse problem, as given previously,

is solved.

The procurement cost of ordering $\underline{z} \geq \underline{0}$ units is assumed to be

equal to a set up cost plus a linear purchase cost

$$c^{T}\underline{z} = (c_1, c_2, \ldots, c_m) \begin{pmatrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_m \end{pmatrix}$$

where $c_i \geq 0$, $i = 1, \ldots, m$, is the unit purchasing cost of item i.

The fixed set up cost is incurred only if an order is made. Thus if

$\underline{z} \geq \underline{0}$ units are ordered, the fixed cost will be a function of $\underline{z}$, say $K(\underline{z})$.

In general $K(\underline{z})$ will take as many different values as the number of

alternative different ordering decisions. For the two commodity problem

$$K(\underline{z}) = \begin{cases} K_1 & \text{if } z_1 > 0 \text{ and } z_2 = 0 \\ K_2 & \text{if } z_1 = 0 \text{ and } z_2 > 0 \\ K & \text{if } z_1 > 0 \text{ and } z_2 > 0 \\ 0 & \text{if } z_1 = 0 \text{ and } z_2 = 0 \end{cases}$$

where K_1, K_2 and K are all nonnegative, and the inequality max (K_1, K_2)

$\leq K \leq K_1 + K_2$ is satisfied. Thus, the procurement cost is given by
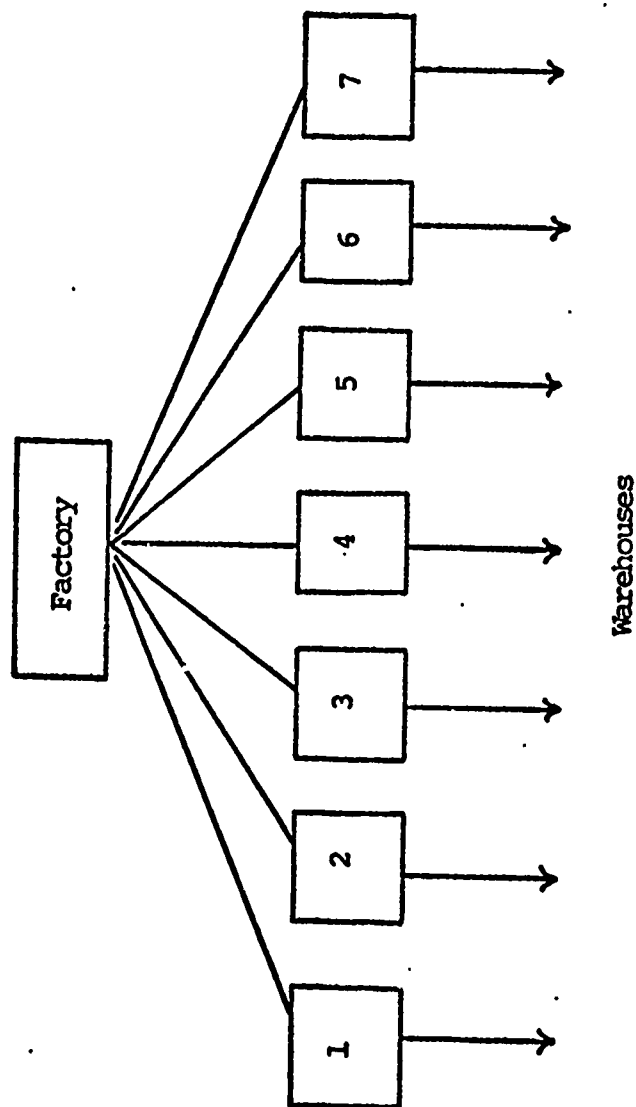
$K(\underline{z}) + c^{T}\underline{z}$

Fig. (1.1). An inventory system with parallel stations.

1.2 Literature Review

We shall first survey the literature on the steady-state or stationary solutions to the one-commodity inventory system operating under a (s,S) policy; next we shall elaborate on the work done on the stationary solutions to the m-commodity inventory system operating under a (σ,S) policy.

For the one-commodity problem we shall consider a dynamic inventory model. At the beginning of each period, a decision is made to order up to S if the stock falls below s, otherwise, nothing is ordered. The demands, $\{D_i\}$, in successive periods (i = 1, 2, ...) are assumed to be described by a sequence of non-negative random variables, independently and identically distributed with a joint density function $\phi_D(\cdot)$. Delivery of orders is immediate and total backlogging of unfilled demands is assumed. The ordering decision in each period is affected by a set up cost K, a linear purchase cost cz where z is the quantity ordered and c the unit cost, a holding cost for carrying inventory, and a shortage cost for not meeting demands. L(x) represents the expected inventory holding and shortage costs for being in stock x at the beginning of the period.

Following the pioneering work of Arrow, Harris, Marchak [1], Karlin [2] studies the steady state for this model operating under a (s,S) policy. With this ordering policy, the sequence of inventory levels X_1, X_2,...., X_n,...., where X_i is the inventory level at the end of the ith period, forms a Markov process. Karlin shows that the distribution of X_n coverages in the sense of distributions to a stationary distribution with the density $f(\cdot)$ given by

$$f(x) = \begin{cases} \dfrac{\psi(S-x)}{1+\Psi(S-s)} & \text{for } s \leq x < S \\[3em] \dfrac{\phi(S-s) + \displaystyle\int_0^{S-s} \psi(t)\,\phi(S-x-t)\,dt}{1+\Psi(S-s)} & \text{for } -\infty \leq x < s \end{cases} \qquad (1.1)$$

where

$$\psi(x) = \sum_{n=1}^{\infty} \phi^{(n)}(x)$$

and

$$\Psi(x) = \sum_{n=1}^{\infty} \Phi^{(n)}(x)$$

$\phi^{(n)}(\cdot)$ and $\Phi^{(n)}(\cdot)$ are the well-known symbolic notations for the nth fold convolution of $\phi(\cdot)$ and $\Phi(\cdot)$, respectively. The stationary distribution is clearly independent of any economic consideration. By imposing the assumed cost structure on the process the total expected cost per period denoted by $g(s, S - s)$ is given as

$$g(s, S-s) = \frac{K + L(S) + \displaystyle\int_0^Q L(S-x)\,\psi(x)\,dx}{1 + \psi(Q)} + c\mu \qquad (1.2)$$

where $Q = S - s$ and $\mu = E(D)$, $E(\cdot)$ is the expectation operator.

Greenberg [5] obtains $\bar{f}_n(\cdot)$, the transient solution for the (s,S) inventory system. Using the generating function approach, the stationary distribution $f(\cdot)$ is given by the standard Abelian theorem

$$f(x) = \lim_{z \to 1} (1 - z) \sum_{n=1}^{\infty} z^n \bar{f}_n(x)$$

which results in (1.1). Iglehart [7] shows that the optimal values of s^* and Q^* minimizing $g(s, S - s)$ are given as the solution to the following equations

$$L'(Q^* + s^*) + \int_0^{Q^*} L'(Q^* + s^* - x) \, \psi(x) \, dx = 0$$

$$L(s^*) = \frac{K + L(Q^* + s^*) + \int_0^{Q^*} L(Q^* + s^* - x) \, \psi(x) \, dx}{1 + \psi(Q^*)}$$

where $L'(\cdot)$ is the derivative of $L(\cdot)$.

Another approach in studying the (s,S) inventory model is followed by Bevazlian [11]. He defines $f_n(x)$ to be the total expected cost of operating the system for n periods when no order is placed initially, i.e., $s \le x < S$, $h_n(x)$ to be the total expected cost of operating the system for n periods when an order is placed initially, i.e., $-\infty < x < s$. Then $f_n(x)$ and $h_n(x)$ satisfy the following functional equations

$$f_n(x) = L(x) + \int_0^{x-s} f_{n-1}(x - t)\, \phi(t)\, dt$$

$$+ \int_{x-s}^{\infty} h_{n-1}(x - t)\, \phi(t)\, dt \qquad s \leq x < S \qquad (n \geq 1)$$

$$h_n(x) = K + c(S - x) + L(S) + \int_0^{S-s} f_{n-1}(S - t)\, \phi(t)\, dt$$

$$+ \int_{S-s}^{\infty} h_{n-1}(S - t)\, \phi(t)\, dt \qquad -\infty < x < s \qquad (n \geq 1)$$

Starting with these basic functional equations and using Howard's approach [6] he deduces the stationary total expected cost per period as given by (1.2). The optimal values for s^* and $Q^* = S^* - s^*$ that minimize the total expected cost per period are given as the solutions to the set of equations

$$M(s^*, Q^*) = K \quad \text{and} \quad \frac{\partial M(s^*, x)}{\partial x}\Bigg|_{x=Q^*} = 0$$

where

$$L\{M(s^*, x)\} = \frac{L\{L(s^*) - L(s^* \cdot x)\}}{1 - L\{\phi(x)\}}$$

the operator $L\{\ \}$ is the Laplace transform operator.

Several computational methods for obtaining values of s^* and Q^* are given and elaborated later by Sivazlian.[1] For the case of gamma distributed demand and linear holding and shortage costs, a dimensional analysis is carried to reduce the number of variables. A numerical inversion technique of the Laplace transform expressions using Gaussian quadrature is used to solve for s^* and Q^*. For large values of Q, Sivazlian [11] and Roberts [3] show that s^* is the solution to the equation

$$\int_{s^*}^{\infty} [1 - \phi(v)] \, dv \approx \frac{u}{h + p - c} \left[\frac{h}{2} \left(1 + \frac{\sigma^2}{\mu^2} \right) + \sqrt{2Kh/\mu} \right]$$

and $Q^* \approx \sqrt{2K\mu/h}$, where μ and σ^2 are the mean and the variance of the random variable D, h is the unit holding cost, and p the unit shortage cost. Sivazlian [11] also shows that for small Q, s^* and Q^* are the solutions for the following approximate system of equations

$$L'(s^*) = -\frac{2K}{Q^*}$$

$$L''(s^*) = \frac{2K}{Q^{*2}} + \frac{2K}{Q^*} \phi(0)$$

It is important to note that an integral equation of the renewal type in one dimension is solved by each of the authors in carrying their analysis. In Scarf's work [2], the density function $f(\cdot)$ as given by (1.1) is a solution of a renewal type integral

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[1]See additional references.

equation. Sivazlian [11] in finding the total expected cost per period solves a renewal type integral equation.

For the case when the demand is discretely distributed, Veinott and Wagner [16] develop the method of stationary or renewal analysis for the computation of an optimal (s,S) policy. The total expected cost per period is given by

$$g(s, S - s) = \frac{K + L(S) + \sum_{j=1}^{S-s} L(S - j)\ \psi(j)}{1 + \Psi(S - s)} + c\mu$$

which is a discrete version of (1.2). The algorithm provided for searching the (s,S) policies to find one that minimizes g(s,S - s) consists of two steps.

Step 1. Determine lower bounds $(\underline{s},\underline{S})$ and upper bounds $(\overline{s},\overline{S})$ on s^* and S^*.

Step 2. Find the collection of all (s,S) policies that minimizes g(s,S - s) over the class of (s,S) policies falling within the bounds defined in Step 1. Any policy in the collection is optimal.

An interval bisection or a Fibonacci search technique is then employed to find the optimal values of s^* and $Q^* = S^* - s^*$. Note that this procedure does not guarantee the optimality of g(s,S - s) since, in general, g(s,S - s) is not a unimodal function in s and Q.

For other work done in the area of one-commodity inventory, the survey articles of Scarf [9] and Veinott [18] should be noted.

Let us now turn to the m-commodity (m $\geq$ 1) problem. As most practical problems in inventory involve more than one product, there has been substantial interest recently in studying the theory of multi-commodity inventory problems. The problem we shall consider is the familiar dynamic inventory system with periodic review [17]. Demands, $\{D_i\}$, for the items over a sequence of successive periods (i = 1,2,...) are assumed to be described by a sequence of non-negative random variables, independently and identically distributed. No specific restrictions will be placed upon the statistical properties of the demands for commodities within a given period; thus, in general, the demand distribution is given by a joint density function $\phi_{\underline{D}}(\underline{t})$ ($\underline{t} \geq \underline{0}$). Complete backlogging and immediate delivery of orders are assumed, and as previously discussed the ordering decision in each period is affected by a single set up cost, a linear ordering cost and holding and shortage costs. Following Sivazlian [13], we define the following:

In E^m, let $\Omega = \{\underline{x} | \underline{x} \leq \underline{S}\}$ where $\underline{x} = (x_1, x_2, \ldots, x_m)$ is the inventory levels of all items prior to a decision and $\underline{S} = (S_1, S_2, \ldots, S_m)$ is a non-negative vector. Let the hypersurface that subdivides the set Ω into the subsets σ and σ^C be defined implicitly by an equation of m variables, $z(\underline{S} - \underline{x}^0) = 0$, $x^0 \varepsilon \Omega$, where the real-valued function $z(\underline{S} - \underline{x})$ has the following properties:

1. $Z(\underline{S - x})$ is defined and continuous for $\forall\ \underline{x}\epsilon\Omega$

2. $Z(\underline{S - x}) < 0\ \forall\ \underline{x}\epsilon\sigma^C$

3. $Z(\underline{S - x}) > 0\ \forall\ \underline{x}\epsilon\sigma$

Define $\Gamma = \{\underline{x}|\underline{x}\epsilon\Omega,\ Z(\underline{S - x}) = 0\}$ and $\omega(\underline{x}) = \{\underline{t}|\underline{t} \leq \underline{x};\ \underline{t},\ \underline{x}\ \epsilon\sigma^C\}$;

The stationary policy considered is to order all items for $\underline{x}\epsilon\sigma^C$. For the case of two commodities, Fig. (1.2) illustrates geometrically the decision regions. It is important to note that for $\underline{x}\not\epsilon\Omega$, the (σ,S) policy is not feasible. However, once a stock level reaches a level $\underline{x}\epsilon\Omega$, the policy is and remains feasible.

With this ordering policy, the sequence of inventory levels $\underline{X}_1,\underline{X}_2,\ldots,\underline{X}_n,\ldots,$ where $\underline{X}_i$ is the inventory level of all items at the beginning of the ith period, forms a Markov process since the demands are independently distributed from one period of time to the next and since the (σ,S) policy depends only on the last state of the process. Sivazlian [10] and [13] studies the stationary behavior of the system for the special case when Γ is admissible, i.e., $\Gamma \cap \Lambda(\underline{x}) = \phi$ (null set) where $\Lambda(\underline{x}) = \{\underline{t}|\underline{x} < \underline{t} < \underline{S},\ \underline{x}\epsilon\sigma^C\}$. If $W(\underline{y})$ denotes the distribution function of the stationary distribution of of the stock levels immediately after a decision, then

$$dW(\underline{y}) = \begin{cases} M & \text{if } \underline{Y} = \underline{S}; \\ \\ \psi(\underline{y}) \; \underline{dy} & \text{if } \underline{y} \; \varepsilon \delta^c \end{cases}$$

where $\delta^c = \sigma^c - \underline{S}$, M is the probability of placing an order, and $\psi(\underline{y})$ satisfies the integral equation

$$\psi(\underline{y}) = M\phi(\underline{S} - \underline{y}) + \int_{\delta^c} \psi(\underline{x}) \; \phi(\underline{x} - \underline{y}) \; \underline{dx}$$
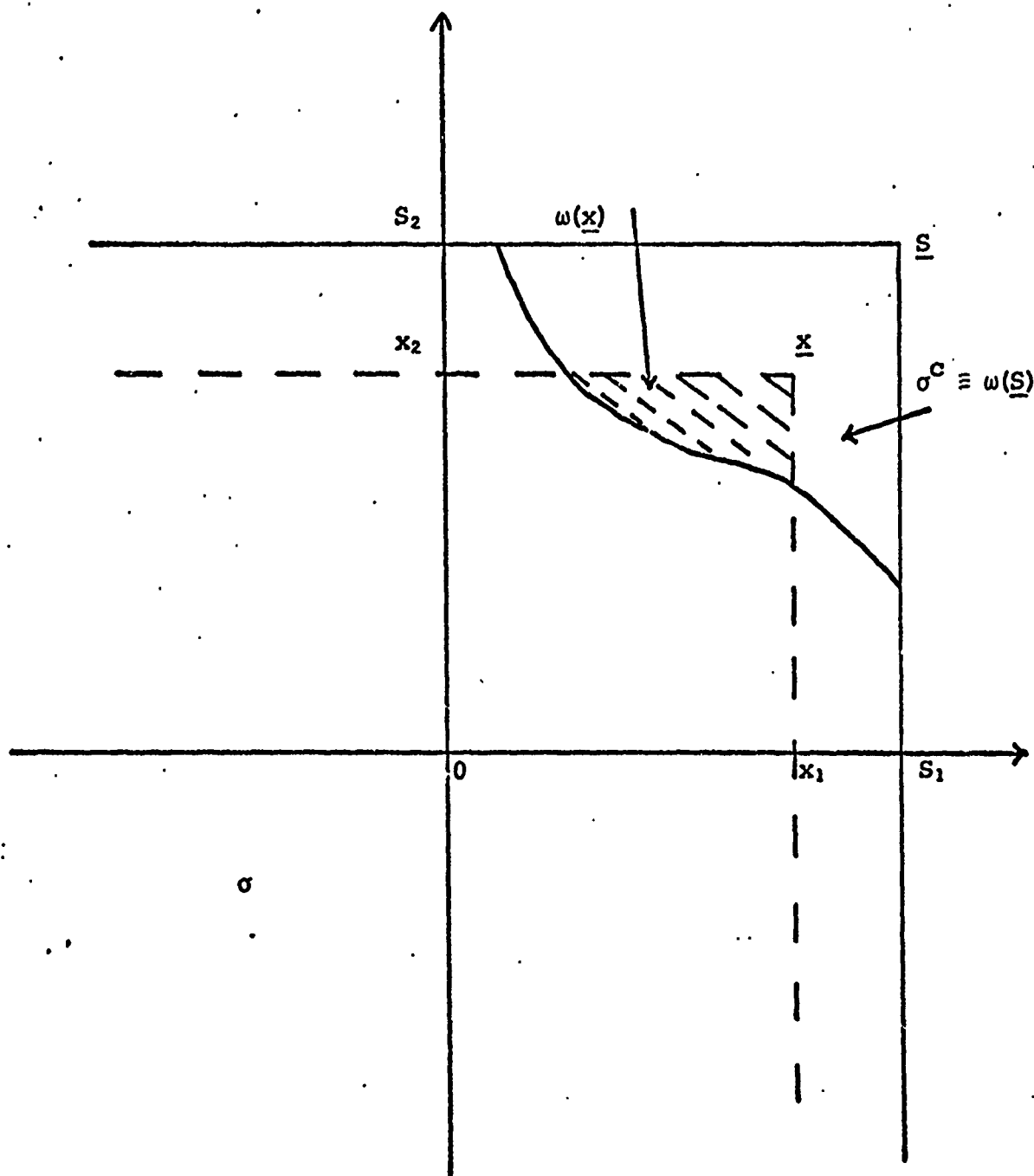
Fig. (1.2). (σ,S) ordering policy for a two-commodity inventory system.

Sivazlian [10] shows that the solution of the integral equation is given by

$$\psi(\underline{y}) = M \sum_{n=1}^{\infty} \phi^{(n)} (\underline{S} - \underline{y})$$

where $\phi^{(n)} (\underline{x})$ is the ordinary n^{th} fold convolution of $\phi(\underline{x})$ with itself and

$$M = \frac{1}{1 + \int_{\underline{S-t\varepsilon\omega}(\underline{S})} \psi(\underline{t}) \, \underline{dt}}$$

The optimal decision variables $S_1, S_2, \ldots, S_m$ and the configuration of the admissible set Γ are derived by minimizing the total expected cost per period given as

$$g(\sigma, \underline{S}) \equiv E(C) = KM + \sum_{i=1}^{m} [c_i E(D_i) + h_i E(Y_i) + (h_i + p_i) E(B_i)] \quad (1.3)$$

where C is the total cost per period; and for items $i = 1, 2, \ldots, m$

c_i = variable cost/unit

h_i = holding cost/unit at the end of a period

p_i = shortage cost/unit at the end of a period

B_i = unfilled demand at the end of a period

Y_i = inventory level at the end of a period

$E(\cdot)$ denotes the expectation operator.

For the two-commodity problem when the demands for the items are independently and exponentially distributed with $\phi_i(t_i) = \lambda_i e^{-\lambda_i t_i}$, $0 \le t_i < \infty$, $\lambda_i > 0$ $(i = 1,2)$, the optimal policy is investigated for the two given classes of admissible Γ

Case 1. Here Γ is the set of points for which

$\lambda_1(S_1 - x_1) + \lambda_2(S_2 - x_2) = a$

where a is an unknown positive

parameter to be determined.

Case 2. Here Γ is the set of points for which

$\max[\lambda_1(S_1 - x_1) - b_1, \lambda_2(S_2 - x_2) - b_2] = 0$

where b_1 and b_2 are unknown positive

parameters to be determined.

For both cases the optimal decision variables are computed. Moreover Sivazlian shows by numerical examples that the joint (σ,S) policy yields a lower operating cost than the case when each item is ordered separately. In another paper Sivazlian [12] considers the multi-commodity control problem operating under a stationary (σ,S) policy. The joint density function of demand for all items in a given period is given by

$$\phi(\underline{t}) = \prod_{i=1}^{m} \phi_i(t_i) \qquad 0 \le t_i < \infty, \qquad (i = 1,2,\ldots m)$$

where $\phi_i'(t_i) = \lambda_i e^{-\lambda_i t_i}$ $(i = 1,2,\ldots,m)$. Γ is admissible and here it is the set of points for which

$$\lambda_1(S_1 - x_1) + \lambda_2(S_2 - x_2) + \ldots + \lambda_m(S_m - x_m) = a \qquad a > 0$$

In this case, let

$$\Delta = \int_{\underline{S}-\underline{t}\varepsilon\omega(\underline{S})} \psi(\underline{t})\ d\underline{t}$$

By generalizing the concept of Dirichlet's multiple integrals, Sivazlian shows that Δ can be expressed as a single integral. Next, similar expressions for $E(Y_i)$ and $E(B_i)$ are obtained. For the case when the variable a takes on large values, explicit analytic solutions for the optimal values of $\underline{S}$ and a are obtained by minimizing (1.3). It is also shown that when m, the number of items, becomes very large and a is finite, the probability of placing an order in any given period tends to 1, and that it is almost optimal to use the "order up to" policy. For the two-commodity problem, Wheeler [20] follows Iglehart's [7] approach for the one-commodity (s,S) model in showing that

$$g(\sigma^*,\underline{S}^*) = \lim_{n\to\infty} \frac{C_n(\underline{x})}{n}$$

where $C_n(\underline{x})$ is the total expected cost of operating the system for n periods, given that $\underline{x}$ is the initial inventory level and an optimal policy is used in each period, and $g(\sigma^*,\underline{S}^*)$ is the optimal stationary total expected cost per period. Following Greenberg's [5] notations

and procedure, Wheeler also attempts to find the transient solutions for the probability distribution of the stock levels.

Wacker[1] formulates and studies the multi-commodity inventory problem with periodic review when a stationary m-dimensional (s,S) policy is followed. Following the work of Roberts [3] he gives analytical formulas for determining the asymptotic reordering region when the set up and shortage costs have large values.

For the case when the demand D in each period is discretely distributed with probability mass function $\phi_D(\underline{x},\underline{t})$, where $\underline{x}$ is the stock level at the beginning of the period, and $\phi_D(\underline{x},\underline{0}) < 1$, Johnson [8] uses the policy iteration method to choose the optimal (σ,S) reorder policy which minimizes the stationary total expected cost per period. Further he provides a computational algorithm for finding bounds and characterizing the optimal policy. The algorithm is a search procedure and only attempts to find the configuration of the optimal policy pointwise.

Finally, the work of Soland [15] is briefly mentioned. In solving some renewal processes in two-dimensions, Soland solves a continuous review inventory problem involving two products for the case when the demand is discretely distributed. In minimizing the total expected cost per cycle (defined as the time between ordering) he concludes that the optimal policy is to order both products at the end of a cycle.

Here again, as in the one-commodity problem, a renewal type integral equation in m-dimensions is solved by each of the authors in carrying their analysis.

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[1]See additional references.

A more explicit definition of the problem is given next.

1.3 Definition of the Problem and an Outline of the Chapters

A multi-commodity inventory system with periodic review operating under a stationary (σ,S) policy is considered. The ordering decision in each period is affected by a single set up cost K, a linear variable ordering cost $\underline{c} = (c_1, c_2,...,c_m)$, and the expected holding and shortage cost function, $L(\underline{x})$, conditional on being in stock level $\underline{x} = (x_1, x_2,...,x_m)$ at the beginning of a period; $L(\underline{x})$ is assumed to be twice differentiable. Demand, $\{D_i\}$, for the items over a sequence of periods $(i = 1,2,...)$, is assumed to be independently and identically distributed continuous non-negative random variables with continuous joint density function $\phi_D(\underline{t})$. Delivery of orders is immediate and total backlogging of unfilled demands is assumed.

Under a stationary (σ,S) policy, as discussed before, either all items will be ordered to bring the inventory level to $\underline{S} = (S_1, S_2, ...,S_m)$ if $\underline{x}$, the inventory level at the beginning of a period prior to making a decision, is in σ (ordering region) or nothing is ordered if $\underline{x} \in \sigma^c$ (not ordering region). The principal concern of this study is to find the optimality condition for (σ,S) policies. This is done by minimizing the expression for the stationary total expected cost per period with respect to the decision variables that characterize the policy being used.

The study begins in Chapter II by introducing some mathematical concepts that are needed in subsequent chapters. First the concept and properties of a generalized convolution operator defined on locally

integrable real-valued functions of m (m > 1) variables are introduced;
next the solution of an integral equation of the renewal type in m
dimensions is studied.

In Chapter III the analysis for deriving an analytical expression
for the stationary total expected cost per period, proceeds in three
steps: (1) the basic functional equations which relate $f_n(\underline{x})$, the
total expected cost of operating the system for n periods when starting
with $\underline{x}\epsilon\sigma^c$, and $h_n(\underline{x})$, the total expected cost of operating the system
for n periods when starting with $\underline{x}\epsilon\sigma$, are derived; (2) for large n it
is shown that $f_n(\underline{x}) = ng + u(\underline{x})$ and $h_n(\underline{x}) = ng + v(x)$; where g is the
stationary total expected cost per period and $u(\underline{x})$, $v(\underline{x})$ are functions
of the initial stock levels; and (3) an analytical expression for
g, which is a function of (σ,S), is derived by solving a renewal
type integral equation that results from the use of the asymptotic
expressions for $f_n(\underline{x})$ and $h_n(\underline{x})$ in the functional equations.

In Chapter IV, the optimality conditions for (σ,S) policies
are studied. The minimization of the expression for the stationary
total expected cost per period with respect to the decision variables
proceeds as follows: First, it is shown that at optimality
$\Gamma^* = \{\underline{x}|\underline{x}\epsilon\Omega;\ L(\underline{x}) - C^* = 0\}$, where C^* is the minimum value of $g(\sigma,\underline{S})$
excluding the variable ordering cost. Then the necessary and suf-
ficient conditions for the existence of proper relative minima of
$g(\sigma,\underline{S})$ at the pair $(\underline{S}^*,C^*)$ are established.

In Chapter V, the study considers the two-commodity inventory
control problem where the demands for each commodity, in a given period,

are independent and exponentially distributed, and where the holding
and shortage costs for each commodity are linear. The equations used
to solve for the optimal policy parameters are analytically determined.
In a numerical example a computer program is developed to determine
S_1^*, S_2^* and C^*.

Finally, in Chapter VI recommendations for future research
are presented.

As compared to other works, in the m-commodity (m > 1)
inventory control problem operating under a stationary (σ,S) policy,
this research is more general, in the sense that no assumptions are
made about the configuration of the ordering region or the admissibility
of Γ. The only restriction on the set $\Gamma = \{\underline{x}|\underline{x}\epsilon\Omega, Z(\underline{S} - \underline{x}) = 0\}$ that
subdivides the set Ω into the ordering region, σ, and the not ordering
region, σ^C, is that the real valued function $Z(\underline{S} - \underline{x})$ is continuous
and defined for all $\underline{x}\epsilon\Omega$. Moreover the ordering regions must be simply-
connected. The ordering regions considered by other researchers, in
the m-commodity problem for continuous demand, are all special cases of
the general case considered here. In E^2, Fig. (1.3) shows geometrically
the configuration of some of the regions that are considered in this
research. Fig. (1.4) shows some ordering regions that will not be
considered. The m-dimensional ordinary convolution operation used by
other researchers in studying the system under consideration is a
special case of the generalized convolution operation introduced in
Chapter II. For the case when the set Γ is admissible, the generalized
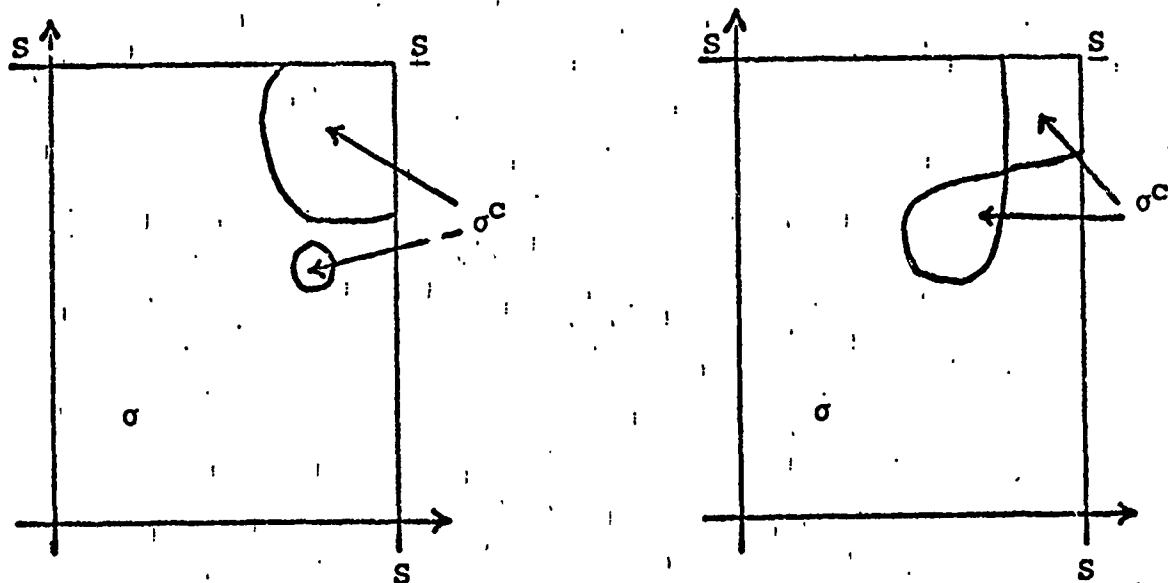convolution of any function reduces to the ordinary convolution and
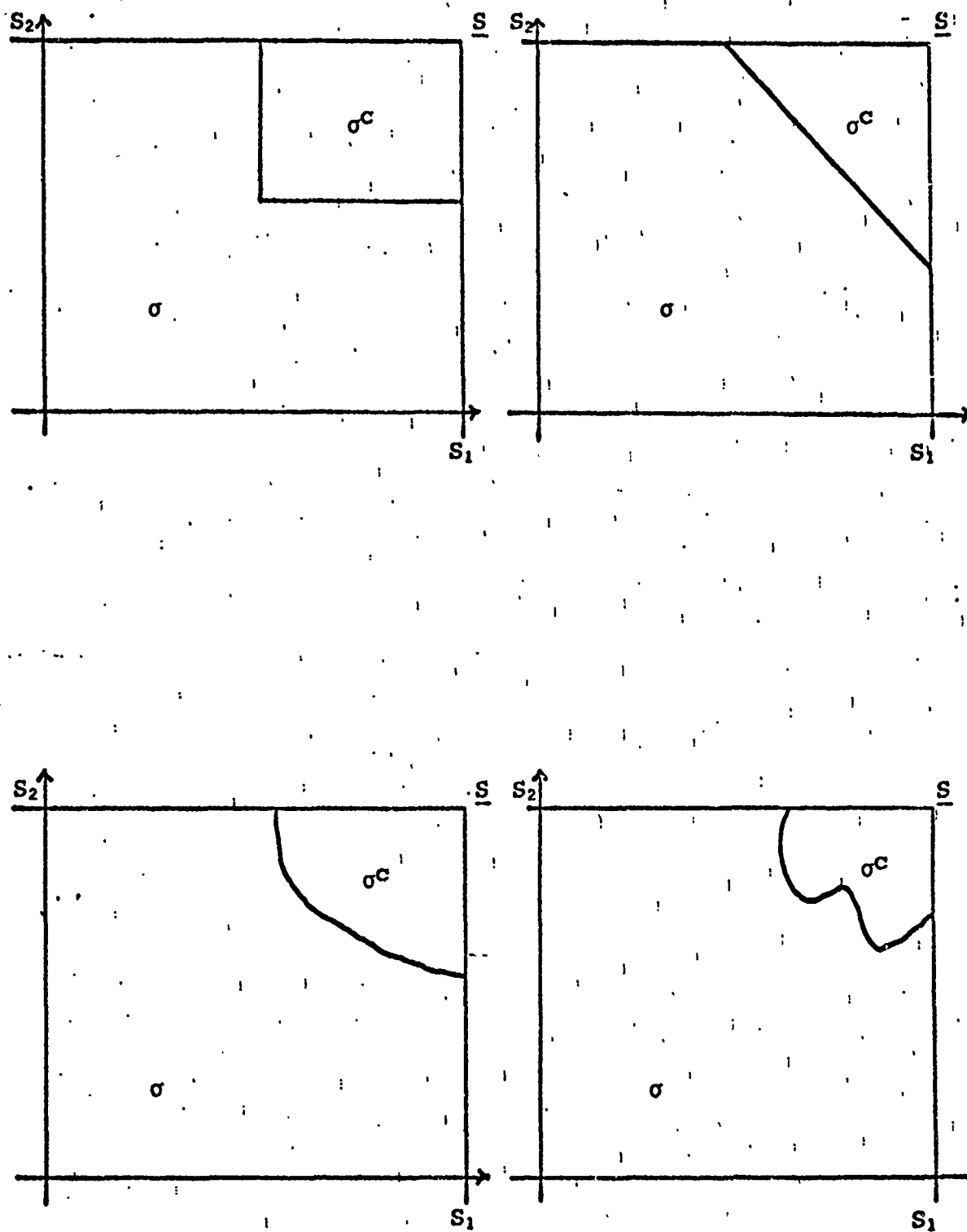
(Fig. (1.3).

Fig. (1.4).

all the generalized convolution operation properties with the commutative law hold. The analysis used in deriving the expression for the stationary total expected cost per period is similar to Sivazlian's [11] approach for the one-commodity problem when operating under a (s,S) policy. The asymptotic expressions for the costs of operating the system are deduced by appealing to the results of Yosida and Kakutan [22]. The set of equations that will be used to determine the optimal parameters of the adopted policy are similar in form to the set of equations used by Sivazlian [11] in determining the optimal values of s^* and $Q^* = S^* - s^*$ for the one commodity problem.

CHAPTER II

MATHEMATICAL PRELIMINARIES

2.1 Introduction

In this chapter a number of mathematical concepts that will be needed in subsequent chapters are discussed. First, the concept and properties of a generalized convolution operator defined on locally integrable real-valued functions in $m(m \geq 1)$ variables are studied and then the solution of a renewal type integral equation is given.

2.2 The Generalized Convolution

Let $\underline{S}$ be a positive vector in a space of $m(m \geq 1)$ dimensions; i.e., $\underline{S} > \underline{0} \Longleftrightarrow S_i > 0$, $i = 1,2,\ldots,m$. In E^m, define the set $\Omega = \{\underline{x} \mid \underline{x} \leq \underline{S}\}$, and let the hypersurface that subdivides the set Ω into the subsets σ and σ^c be defined implicitly by an equation of m variables; $Z(\underline{S}-\underline{x}^0) = 0$, $\underline{x}^0 \varepsilon \Omega$, where the real-valued function $Z(\underline{S}-\underline{x})$ have the following properties:

1. $Z(\underline{S}-\underline{x})$ is defined and continuous for $\forall \underline{x} \varepsilon \Omega$;

2. $Z(\underline{S}-\underline{x}) < 0$ $\forall \underline{x} \varepsilon \sigma^c$

3. $Z(\underline{S}-\underline{x}) > 0$ $\forall \underline{x} \varepsilon \sigma$

Next, define the sets:

(a) $\omega(\underline{S}) = \{\underline{x} \mid \underline{x} \varepsilon \Omega; Z(\underline{S}-\underline{x}) < 0\}$;

(b) $\omega(\underline{x}) = \{\underline{t} \mid \underline{t}, \underline{x} \varepsilon \omega(\underline{S}); \underline{t} \leq \underline{x}\}$;

(c) $\Gamma = \{\underline{x} \mid \underline{x} \varepsilon \Omega; Z(\underline{S}-\underline{x}) = 0\}$;

(d) $\omega'(\underline{t},\underline{x}) = \{\underline{v} \mid \underline{t} < \underline{v} < \underline{x}; \underline{t}, \underline{v} \varepsilon \omega(\underline{x})\}$;

(e) $r''(\underline{x}) = \{\underline{t} \mid \underline{t} \leq \underline{x};\ \underline{x}\varepsilon\omega(\underline{S})\};$

(f) $r'(\underline{x}) = r''(\underline{x}) - \omega(\underline{x})$

Let us make the transformation $\underline{u} = \underline{x-t}$, $\underline{t}\varepsilon\omega(\underline{x})$, then the set $R(\underline{x})$ is the image of the set $\omega(\underline{x})$, Γ_0 is the image of Γ, and $R_1(\underline{x-t})$ is the image of $\omega'(\underline{t},\underline{x})$. In the particular case $m = 2$, Fig. (2.1) illustrates geometrically the effects of the transformation on the above defined sets.

For $\underline{u}\varepsilon R(\underline{x})$, define the set $R_1(\underline{u}) = \{\underline{t} \mid \underline{0} \leq \underline{t} \leq \underline{u};\ \underline{t}\varepsilon R(\underline{x})\}$. Consider the set of functions D, $f(\underline{t})\varepsilon D$ is a real valued function of m variables $(t_1, t_2, \ldots, t_m)$ for which we have

1. $f(\underline{t}) \equiv 0$ $\qquad\qquad$ for $\underline{t} < \underline{0}$

2. $f(\underline{t})$ is locally integrable on $R_1(\underline{u})$

$\Big\}$ (2.1)

On this set of functions, addition and g-convolution operations are defined as follows:

1. Addition $(f+g)(\underline{t}) \overset{Def}{=} f(\underline{t}) + g(\underline{t})$;

2. g-convolution $(f*g)(\underline{u}) \overset{Def}{=} \int_{\underline{v}\varepsilon R_1(\underline{u})} f(\underline{u-v})\ g(\underline{v})\ \underline{dv}$

then the following relations are satisfied

1. $f(\underline{t})$, $g(\underline{t})\varepsilon D \implies f(\underline{t}) + g(\underline{t})\varepsilon D$

2. $f(\underline{t}) + g(\underline{t}) = g(\underline{t}) + f(\underline{t})$ (Commutative Law)

3. $\{f(\underline{t}) + g(\underline{t})\} + h(\underline{t}) = f(\underline{t}) + \{g(\underline{t}) + h(\underline{t})\}$ (Associative Law)

where the equality sign is used in the ordinary sense. And the set of functions D is an abelian group with respect to addition in which the zero element is the function $f(\underline{t}) \equiv 0$ and the inverse of the function $f(\underline{t})$ is $- f(\underline{t})$.
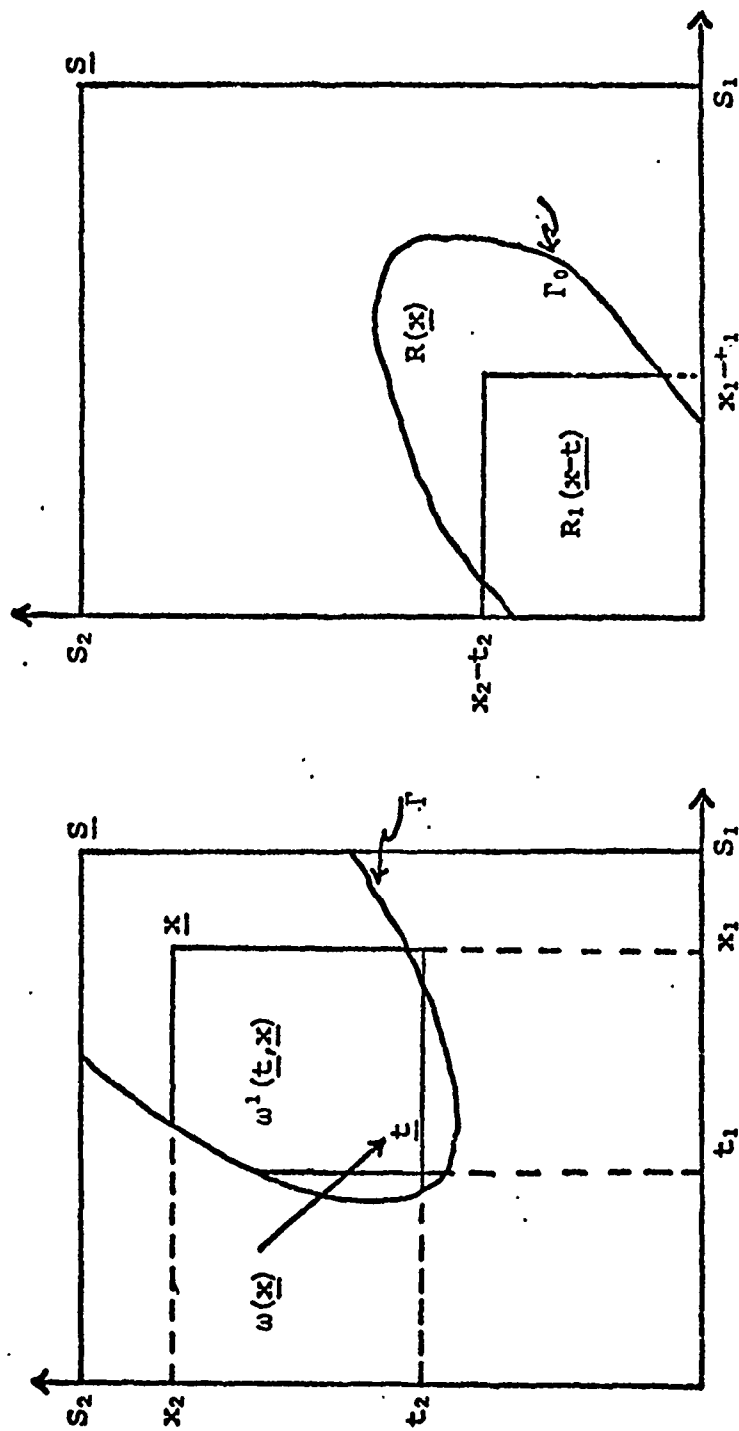
Fig. (2.1).

Properties of q-Convolution

1. Closure Law

If $f(t)$, $g(\underline{t}) \epsilon D$, then $h(\underline{u})$, given by

$$h(\underline{u}) = \int_{\underline{t} \epsilon R_1(\underline{u})} f(\underline{u-t}) \ g(\underline{t}) \ \underline{dt}$$

belongs to D.

Proof: $h(\underline{u})$ is locally integrable on $R_1(\underline{u})$ since both

functions f and g are locally integrable on $R_1(\underline{u})$.

Moreover $h(\underline{u})$ is identically zero for $\underline{u} < \underline{0}$.

2. Distributive Law

If f, g, h, ϵ D then

$$[f * (g+h)] \ (\underline{u}) = (f * g)(\underline{u}) + (f * h)(\underline{u})$$

Proof: By definition

$$[f * (g + h)] \ (\underline{u}) = \int_{\underline{y} \epsilon R_1(\underline{u})} f(\underline{u-y}) \ [g(\underline{y}) + h(\underline{y})] \ \underline{dy}$$

$$= \int_{\underline{y} \epsilon R_1(\underline{u})} f(\underline{u-y}) \ g(\underline{y}) \ \underline{dy}$$

$$+ \int_{\underline{y} \epsilon R_1(\underline{u})} f(\underline{u-y}) \ h(\underline{y}) \ \underline{dy}$$

$$= (f * g)(\underline{u}) + (f * h)(\underline{u})$$

3. Associative Law

If f, g, ϵ D then $[f * (g * h)] \ (\underline{u}) = [(f * g) * h] \ (\underline{u})$

Proof: By definition

$$[f * (g * h)] \ (\underline{u}) = \int_{\underline{v} \epsilon R_1(\underline{u})} f(\underline{u-v}) \int_{\underline{y} \epsilon R_1(\underline{v})} g(\underline{v-y}) \ h(\underline{y}) \ \underline{dy} \ \underline{dv}$$

$$= \int_{W_1} \int f(\underline{u-v}) \ g(\underline{v-y}) \ h(\underline{y}) \ \underline{dy} \ \underline{dv} \qquad (2.2)$$

where $W_1 = \{(\underline{v}, \underline{y}) \mid \underline{v}\epsilon R_1(\underline{u}), \underline{y}\epsilon R_1(\underline{v})\}$ which is equivalent

to the set W_2 defined as

$$W_2 = \{(\underline{v}, \underline{y}) \mid \underline{y} \le \underline{v} \le \underline{u}; \underline{v}, \underline{y}\epsilon R_1(\underline{u})\}$$

Also by definition we can write

$$[(f * g) * h](\underline{u}) = \int\limits_{\underline{y}\epsilon R_1(\underline{u})} [\int\limits_{\underline{t}\epsilon R_1(\underline{u}-\underline{y})} f(\underline{u}-\underline{y}-\underline{t})g(\underline{t}) \, d\underline{t}] \, h(\underline{y}) \, d\underline{y} \qquad (2.3)$$

Let $\underline{t} = \underline{v} - \underline{y}$, then (2.3) becomes

$$[(f * g) * h](\underline{u}) = \int\limits_{\underline{y}\epsilon R_1(\underline{u})} [\int\limits_{\underline{v}-\underline{y}\epsilon R_1(\underline{u}-\underline{y})} f(\underline{u}-\underline{v}) \, g(\underline{v}-\underline{y}) \, d\underline{v}] \, h(\underline{y}) \, d\underline{y}$$

Since $g(\underline{v}-\underline{y}) = 0$ for $\underline{v}-\underline{y} < \underline{0}$, $\underline{v}-\underline{y}\epsilon R_1(\underline{u}-\underline{v})$ implies that $\underline{v}\epsilon R^1(\underline{y},\underline{u})$,

where $R^1(\underline{y},\underline{u}) = \{\underline{v} \mid \underline{y} \le \underline{v} \le \underline{u}, \underline{y}\epsilon R_1(\underline{u})\}$. Hence we have

$$[(f * g) * h](\underline{u}) = \int\limits_{\underline{y}\epsilon R_1(\underline{u})} [\int\limits_{\underline{v}\epsilon R^1(\underline{y},\underline{u})} f(\underline{u}-\underline{v}) \, g(\underline{v}-\underline{y}) \, d\underline{v}] \, h(\underline{y}) \, d\underline{y}$$

$$= \int\int\limits_{W_3} f(\underline{u}-\underline{v}) \, g(\underline{v}-\underline{y}) \, h(\underline{y}) \, d\underline{v} \, d\underline{y} \qquad (2.4)$$

where $W_3 = \{(\underline{v}, \underline{y}) \mid \underline{y} \le \underline{v} \le \underline{u}; \underline{v}, \underline{y}\epsilon R_1(\underline{u})\}$. In E^2,
Fig. (2.2) illustrates geometrically the sets appearing
in (2.2), (2.3), and (2.4).

The sets W_3 and W_2 as defined are equivalent, and that in
turn implies W_1 and W_3 are equivalent. Since the integrals given
by (2.2) and (2.4) exist, then by Fubini's theorem the iterated
integrals are equal. Therefore, for each $\underline{u}\epsilon R(\underline{S})$

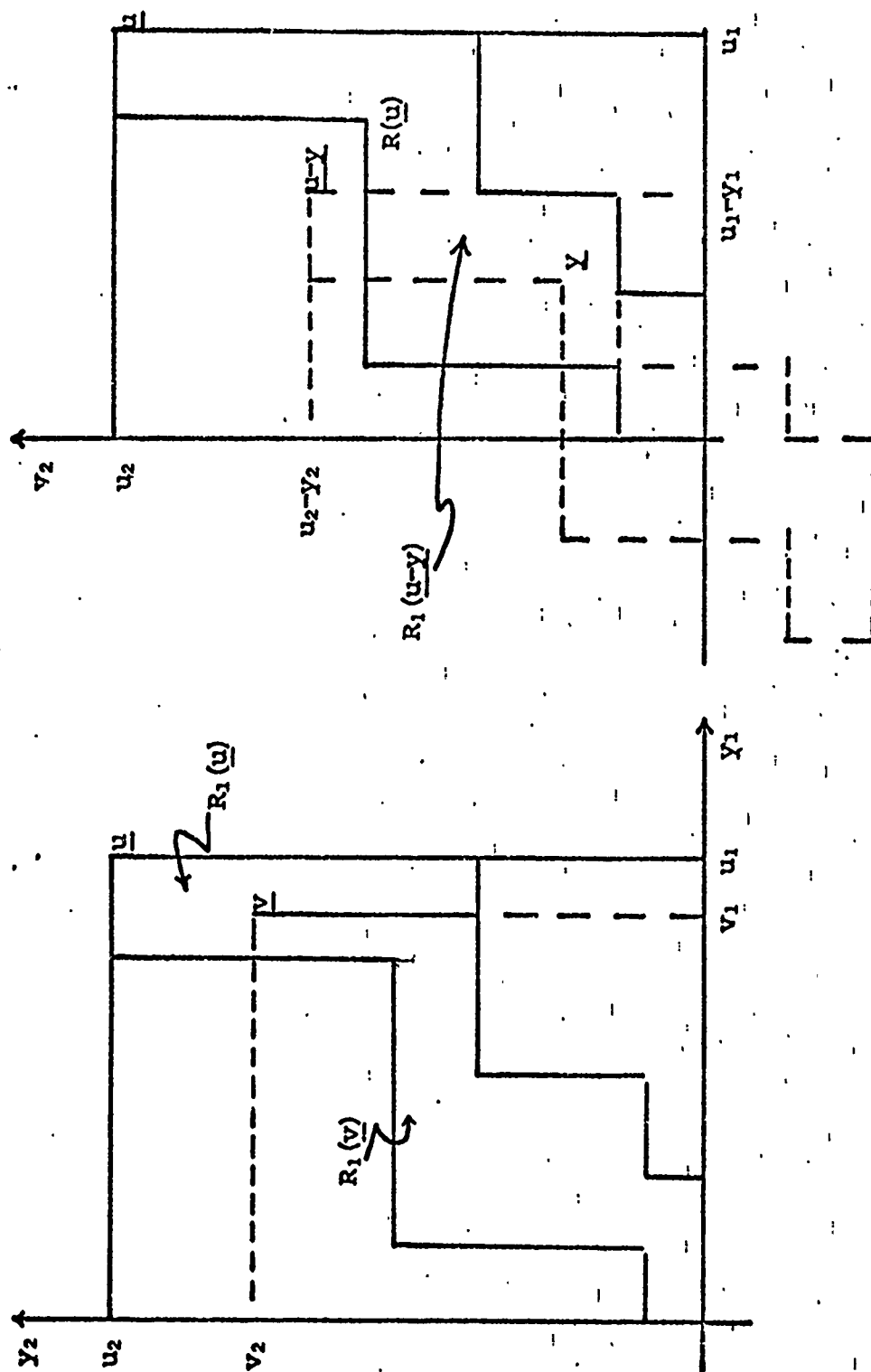$$[f * (g * h)](\underline{u}) = [(f * g) * h](\underline{u})$$

Fig. (2.2).

4. **Commutative Law:**

We observe that in general, if $f, g \in D$, then

$$(f^*g)(\underline{u}) \neq (g^*f)(\underline{u})$$

By definition

$$(f^*g)(\underline{u}) = \int_{\underline{t} \in R_1(\underline{u})} f(\underline{u} - \underline{t}) \, g(\underline{t}) \, d\underline{t}$$

Let $\underline{u} - \underline{t} = \underline{v}$; then

$$(f^*g)(\underline{u}) = \int_{\underline{v} \in \bar{R}_1(\underline{u})} f(\underline{v}) \, g(\underline{u}-\underline{v}) \, d\underline{v} \neq (g^*f)(\underline{u})$$

where $\bar{R}_1(\underline{u}) = \{\underline{v} \mid \underline{u}-\underline{v} \in R_1(\underline{u})\}$.

<u>Note:</u> the g-convolution of $f \in D$ with itself commutes.

Moreover, if we define for $n = 1, 2, \ldots$

$$f_{(1)}(\underline{u}) = f(\underline{u})$$

$$f_{(n)}(\underline{u}) = (f_{(n-1)}{}^*f) = \int_{\underline{t} \in R_1(\underline{u})} f_{(n-1)}(\underline{u}-\underline{t}) \, f(\underline{t}) \, d\underline{t} \quad (n \geq 2)$$

then from the g-convolution associativity law we can write

$$f_{(n)}(\underline{u}) = \int_{\underline{t} \in R_1(\underline{u})} f_{(n-1)}(\underline{u}-\underline{t}) \, f(\underline{t}) \, d\underline{t} = \int_{\underline{t} \in R_1(\underline{u})} f(\underline{u}-\underline{t}) \, f_{(n-1)}(\underline{t}) \, d\underline{t} \quad (n \geq 2)$$

If we let $\underline{u}-\underline{t} = \underline{v}$; then $R_1(\underline{u})$ is mapped into $\bar{R}_1(\underline{u})$ and we can write

$$f_{(n)}(\underline{u}) = \int_{\underline{t} \in R_1(\underline{u})} f_{(n-1)}(\underline{u}-\underline{t}) \, f(\underline{t}) \, d\underline{t} = \int_{\underline{t} \in \bar{R}_1(\underline{u})} f(\underline{u}-\underline{t}) \, f_{(n-1)}(\underline{t}) \, d\underline{t}$$

$$= \int_{\underline{t} \in R_1(\underline{u})} f(\underline{u}-\underline{t}) \, f_{(n-1)}(\underline{t}) \, d\underline{t} = \int_{\underline{t} \in \bar{R}_1(\underline{u})} f_{(n-1)}(\underline{u}-\underline{t}) \, f(\underline{t}) \, d\underline{t} \quad (n \geq 2)$$

If E^2, Fig. (2.3) illustrates geometrically the sets

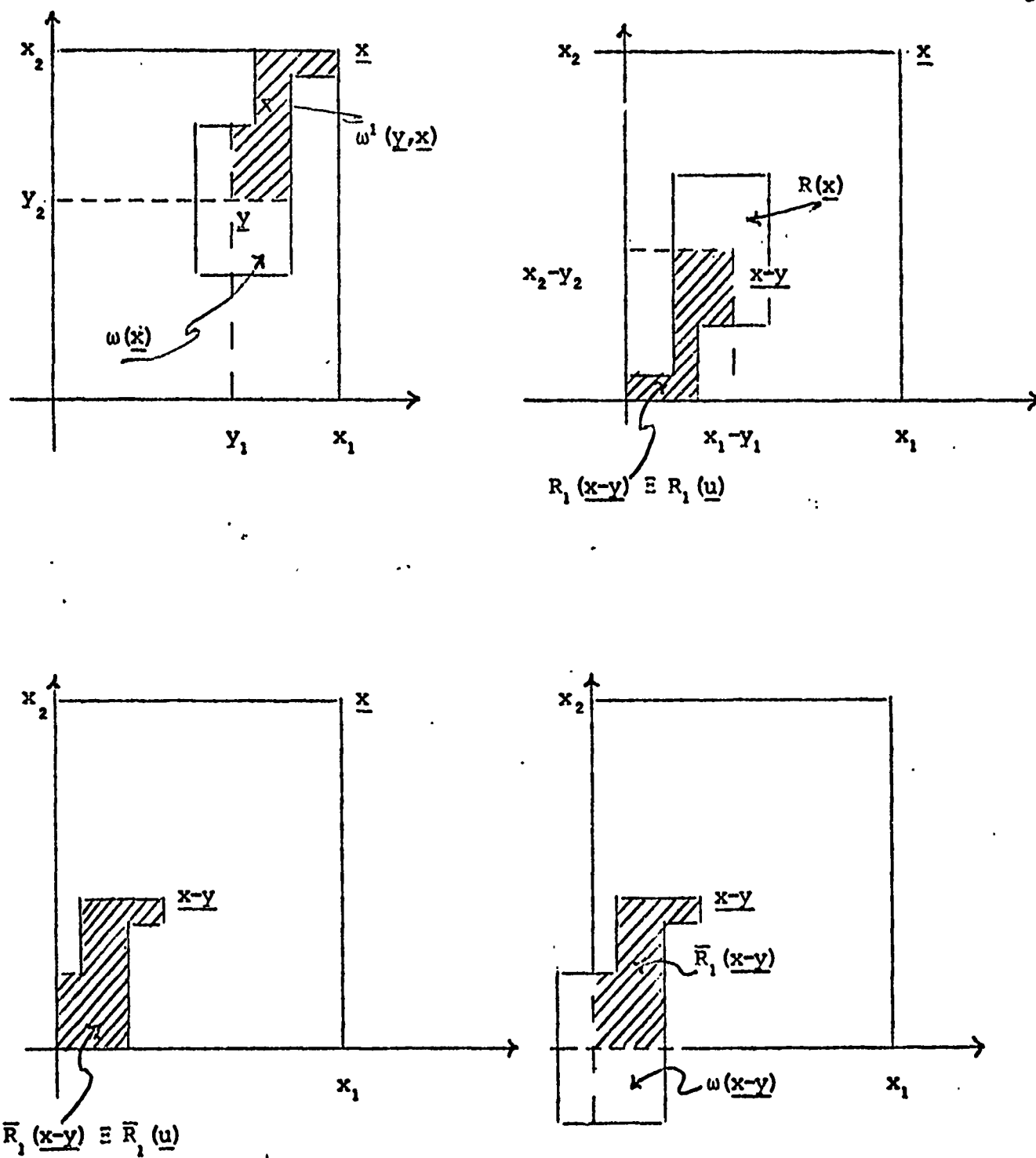$R_1(\underline{u})$ and $\bar{R}_1(\underline{u})$.

Fig. (2.3)

Henceforth we conclude that the set of functions D defined by (2.1) forms a non-commutative ring with respect to addition and convolution operations.

Remark 2.1:

It is important to note that, in the case when for all $\underline{u}\epsilon R(\underline{S})$, $R_1(\underline{u}) = \{\underline{t} \mid \underline{0} \leq \underline{t} \leq \underline{u}\}$, the ordinary convolution in m (m $\geq$ 1) dimensions is defined and all the previous properties with the Commutative Law hold. Thus, in a sense, the ordinary convolution is a special case of the g-convolution. It may also be noted that for the case when Γ_0 is admissible the g-convolution of any function reduces to the ordinary convolution.

Remark 2.2:

Let $\phi(\underline{x})$ be a locally integrable density function of the random variable $\underline{D}$. For every $\underline{x}\epsilon\omega(\underline{S})$, define

$$\Phi_{(1)}(\underline{x}) = \int\limits_{\underline{x}-\underline{t}\epsilon\omega(\underline{x})} \phi(\underline{t}) \, d\underline{t}$$

$$\Phi^{(1)}(\underline{x}) = \int\limits_{\underline{0}}^{\underline{x}} \phi(\underline{t}) \, d\underline{t}$$

and for n $\geq$ 2

$$\Phi_{(n)}(\underline{x}) = \int\limits_{\underline{x}-\underline{t}\epsilon\omega(\underline{x})} \Phi_{(n-1)}(\underline{x}-\underline{t}) \, \phi(\underline{t}) \, d\underline{t}$$

$$\Phi^{(n)}(\underline{x}) = \int\limits_{\underline{0}}^{\underline{x}} \phi^{(n-1)}(\underline{x}-\underline{t}) \, \phi(\underline{t}) \, d\underline{t} .$$

From the above definitions and the definition of the set $\omega(\underline{x})$, we have by induction

$$\Phi_{(n)}(\underline{x}) \le \Phi^{(n)}(\underline{x}) \qquad (n \ge 1)$$

Thus,

$$\sum_{n=1}^{\infty} \Phi_{(n)}(\underline{x}) \le \sum_{n=1}^{\infty} \Phi^{(n)}(\underline{x})$$

By definition

$$\Phi(\underline{x}) = \int_{\underline{0}}^{\underline{x}} \phi(\underline{t}) \, d\underline{t} \le \int_0^{x_i} \int_0^{\infty}\ldots\int_0^{\infty} \phi(t_1,\ldots,t_i,\ldots,t_m) \, dt_1\ldots dt_i\ldots dt_m$$

$$= \int_0^{x_i} \phi_i(t_i) \, dt_i = \Phi_i(x_i) \qquad (i=1,2,\ldots,m)$$

where $\Phi_i(x_i)$ is the marginal distribution of the random variable D_i with density function $\phi_i(x_i)$. If we denote by $\Phi_i^{(n)}(x_i)$ the n-fold ordinary convolution of $\Phi_i(x_i)$, then for $n = 1,2,\ldots$

$$\Phi^{(1)}(\underline{x}) \le \Phi_i^{(1)}(x_i)$$

$$\Phi^{(2)}(\underline{x}) = \int_{\underline{0}}^{\underline{x}} \Phi^{(1)}(x-t) \, \phi(\underline{t}) \, d\underline{t}$$

$$\le \int_0^{x_i} \Phi_i^{(1)}(x_i-t_i) \, dt_i \int_0^{\infty}\ldots\int_0^{\infty} \phi(t_1\ldots t_j\ldots t_m) \, dt_1\ldots dt_m$$

$$= \int_0^{x_i} \Phi_i^{(1)}(x_i - t_i) \, \phi_i(t_i) \, dt_i = \Phi_i^{(2)}(x_i)$$

and

$$\Phi^{(n)}(\underline{x}) \leq \Phi_i^{(n)}(x_i) \qquad (n \geq 1)$$

Henceforth,

$$\sum_{n=1}^{\infty} \phi_{(n)}(\underline{x}) \leq \sum_{n=1}^{\infty} \Phi^{(n)}(\underline{x}) \leq \sum_{n=1}^{\infty} \Phi_i^{(n)}(x_i)$$

From renewal theory in one dimension [2],

$\sum_{n=1}^{\infty} \Phi_i^{(n)}(x_i)$ converges uniformly to $\psi_i(x_i)$, which is the expected

number of renewals that have occurred in the close interval $[0, x_i]$.

Therefore, the series being dominated by a uniformly convergent

series is itself uniformly convergent for every $\underline{x} \varepsilon \omega(\underline{S})$.

2.3 On the Solution of an Integral Equation

Consider the set of functions F, $f(\underline{t}) \varepsilon F$ is a real valued

function of m variables $(t_1, t_2, \ldots, t_m)$ for which we have

1. $f(\underline{t}) \equiv 0$ for $\underline{t} < \underline{0}$

2. $f(\underline{t})$ is locally integrable on $\omega(\underline{S})$

Lemma 2.1: Let f, $g \in F$ and h be a locally integrable function on $\omega(\underline{S})$. For $\underline{x} \in \omega(\underline{S})$, if we denote by

$$a(\underline{x}) = \int_{\underline{t} \in \omega(\underline{x})} f(\underline{x-t}) \left[\int_{\underline{y} \in \omega(\underline{t})} g(\underline{t-y}) \, h(\underline{y}) \, d\underline{y} \right] d\underline{t} \qquad (2.5)$$

$$b(\underline{x}) = \int_{\underline{y} \in \omega(\underline{x})} \left[\int_{\underline{v} \in \omega(\underline{x-y})} f(\underline{x-y-v}) \, g(\underline{v}) \, d\underline{v} \right] h(\underline{y}) \, d\underline{y} \qquad (2.6)$$

$$C(\underline{x}) = \int_{\underline{y} \in \omega(\underline{x})} \left[\int_{\underline{v} \in \overline{R}_1(\underline{x-y})} f(\underline{x-y-v}) \, g(\underline{v}) \, d\underline{v} \right] h(\underline{y}) \, d\underline{y} \qquad (2.7)$$

then,

$$a(\underline{x}) = b(\underline{x}) = C(\underline{x})$$

Proof: In (2.5) $g(\underline{t-y}) = 0$ for $\underline{y} > \underline{t}$. Hence we can write

$$a(\underline{x}) = \int_{\underline{t} \in \omega(\underline{x})} f(\underline{x-t}) \left[\int_{\underline{y} \in \omega(\underline{x})} g(\underline{t-y}) \, h(\underline{y}) \, d\underline{y} \right] d\underline{t} \qquad (2.8)$$

Let $\underline{t} = \underline{v+y}$, then (2.6) becomes

$$b(\underline{x}) = \int_{\underline{y} \in \omega(\underline{x})} \left[\int_{\underline{t-y} \in \omega(\underline{x-y})} f(\underline{x-t}) \, g(\underline{t-y}) \, d\underline{t} \right] h(\underline{y}) \, d\underline{y}$$

Now since $\underline{t-y} \in \omega(\underline{x-y})$ implies that $\underline{t} \in \omega(\underline{x})$, we can write

$$b(\underline{x}) = \int_{\underline{y} \in \omega(\underline{x})} \left[\int_{\underline{t} \in \omega(\underline{x})} f(\underline{x-t}) \, g(\underline{t-y}) \, d\underline{t} \right] h(\underline{y}) \, d\underline{y} \qquad (2.9)$$

The integrals given by (2.8) and (2.9) exist. Therefore, by Fubini's theorem the iterated integrals are equal. Hence for $\underline{x} \varepsilon \omega(\underline{S})$ we have

$$a(\underline{x}) = b(\underline{x})$$

From (2.6) since $g(\underline{v}) = 0$ for $\underline{v} \notin \overline{R}_1 (\underline{x-y}) \varepsilon \omega (\underline{x-y})$, we have

$$b(\underline{x}) = \int_{\underline{y} \varepsilon \omega(\underline{x})} [\int_{\underline{v} \varepsilon \overline{R}_1 (\underline{x-v})} f(\underline{x-y-v}) g(\underline{v}) \, d\underline{v}] h(\underline{y}) \, d\underline{y} = C(\underline{x})$$

And this completes the proof of the lemma.

Note: Let $\underline{x-y-v} = \underline{t}$, then from (2.7)

$$C(\underline{x}) = \int_{\underline{y} \varepsilon \omega(\underline{x})} [\int_{\underline{t} \varepsilon R_1 (\underline{x-y})} f(\underline{t}) \, g(\underline{x-y-t}) \, d\underline{t}] h(\underline{y}) \, d\underline{y}$$

Remark 2.3: Let $\phi(\underline{t})$ be a locally integrable density function of the random variable $\underline{D}$, and let $\phi_{(n)}(\underline{t})$, $n \geq 1$, be the n-fold g-convolution of $\phi(\underline{t})$ with itself. From the g-convolution properties we have

$$\int_{\underline{v} \varepsilon \overline{R}_1 (\underline{x-y})} \phi(\underline{x-y-v}) \, \phi_{(n-1)}(\underline{v}) \, d\underline{v} = \int_{\underline{v} \varepsilon R_1 (\underline{x-y})} \phi_{(n-1)}(\underline{x-y-v}) \, \phi(\underline{v}) \, d\underline{v}$$

$$= \phi_{(n)}(\underline{x-y}) \quad (n \geq 1)$$

Hence from Lemma 2.1

$$\int_{\underline{t} \varepsilon \omega(\underline{x})} \phi(\underline{x-t}) [\int_{\underline{y} \varepsilon \omega(\underline{t})} \phi_{(n-1)}(\underline{t-y}) h(\underline{y}) \, d\underline{y}] \, d\underline{t}$$

$$= \int_{\underline{y} \varepsilon \omega(\underline{x})} \phi_{(n)}(\underline{x-y}) h(\underline{y}) \, d\underline{y}$$

Similarly from the g-convolution properties and Lemma 2.1 we have

$$\int_{\underline{t}\epsilon\omega(\underline{x})} \phi_{(n-1)} \underline{(x-t)} \left[\int_{\underline{y}\epsilon\omega(\underline{t})} \phi \underline{(t-y)} \ h(\underline{y}) \ \underline{dy} \right] \underline{dt}$$

$$= \int_{\underline{y}\epsilon\omega(\underline{x})} \phi_{(n)} \underline{(x-y)} \ h(\underline{y}) \ \underline{dy}$$

In E^2, Fig. (2.4) illustrates geometrically the sets $\omega(\underline{x})$, $\omega\underline{(x-y)}$, and $\bar{R}_1\underline{(x-y)}$.

Theorem 2.1: Let $\phi(\underline{x})$ be a locally integrable density function of the random variable $\underline{D}$. Then, for every $\underline{x}\epsilon\omega(\underline{S})$ and $\|A(\underline{x})\| < \infty$ there exists a unique solution to

$$U(\underline{x}) = A(\underline{x}) + \int_{\underline{t}\epsilon\omega(\underline{x})} \phi\underline{(x-t)} \ U(\underline{t}) \ \underline{dt} \tag{2.10}$$

given by

$$\bar{U}(\underline{x}) = A(\underline{x}) + \int_{\underline{t}\epsilon\omega(\underline{x})} \psi\underline{(x-t)} \ A(\underline{t}) \ \underline{dt} \tag{2.11}$$

where $\psi\underline{(x-t)} = \sum_{n=1}^{\infty} \phi_{(n)} \underline{(x-t)}$, $\phi_{(n)} \underline{(x-t)}$ is the n-fold g-convolution of $\phi\underline{(x-t)}$ with itself, and $\|A(\underline{x})\| = \underset{\underline{x}\epsilon\omega(\underline{S})}{\text{Sup}} |A(\underline{x})|$.

Proof: To prove solution existence and uniqueness, successive approximation method is used. Consider the sequence of functions defined inductively as follows:

$$U_0(\underline{x}) = A(\underline{x})$$

$$U_1(\underline{x}) = A(\underline{x}) + \int_{\underline{t}\epsilon\omega(\underline{x})} U_0(\underline{t}) \ \phi\underline{(x-t)} \ \underline{dt}$$

Fig. (2.4)

$$\vdots$$

and $\qquad U_{n+1}(\underline{x}) = A(\underline{x}) + \int_{\underline{t} \varepsilon \omega(\underline{x})} U_n(\underline{t}) \, \phi(\underline{x-t}) \, \underline{dt} \qquad (n \geq 0)$
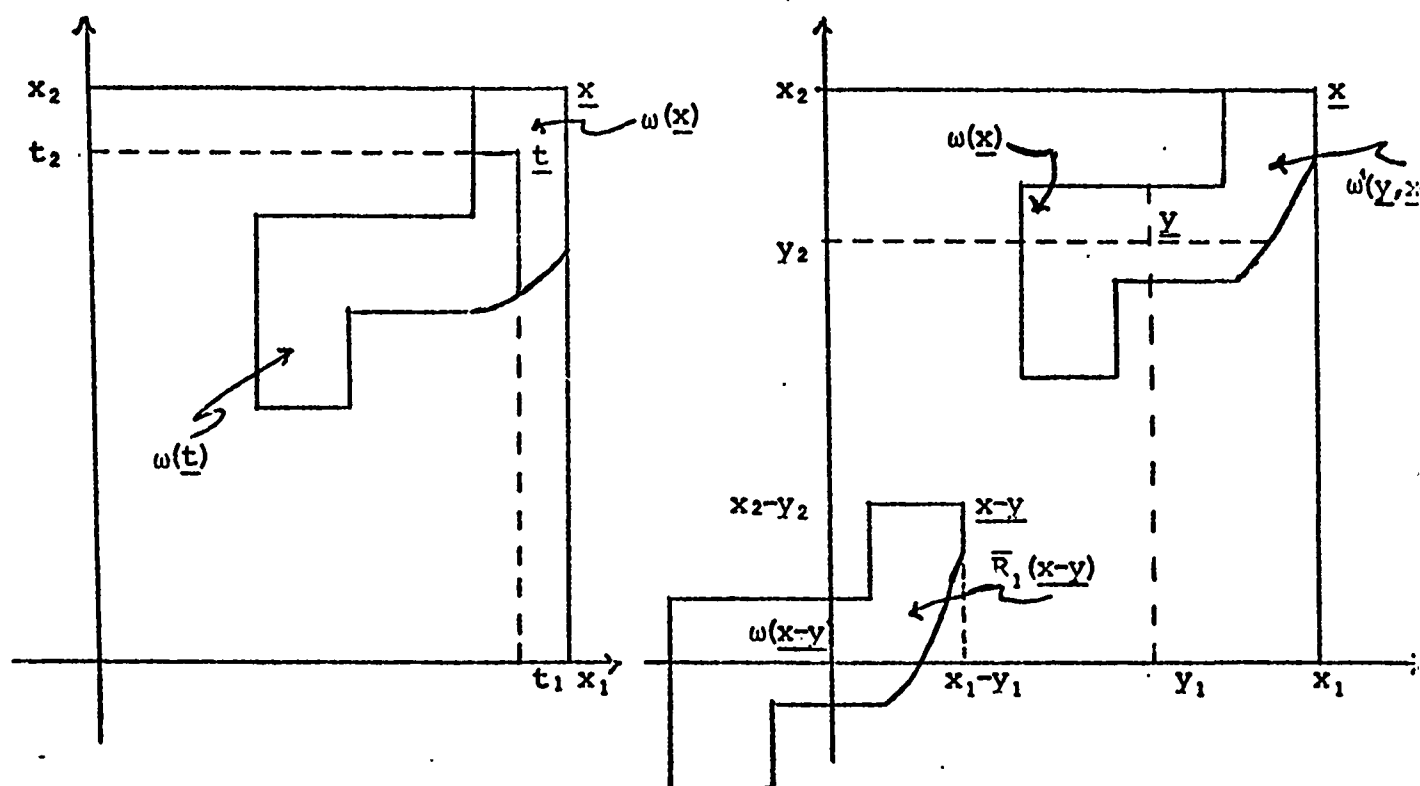
where the subscripts refer to the inductive process. Let A_1 denotes the collection of all such functions. A_1 forms a Banach space in which the norm of an element is defined by

$$\| U(\underline{x}) \| = \underset{\underline{x} \varepsilon \omega(\underline{S})}{\text{Sup}} |U(\underline{x})|$$

To proceed in the proof, first we wish to show that the sequence of functions $\{U_n(\underline{x})\}$ converges to a function $\bar{U}(\underline{x})$ for all $\underline{x} \varepsilon \omega(\underline{S})$. Next, we shall show that this function $\bar{U}(\underline{x})$ satisfies (2.10), and finally that $\bar{U}(\underline{x})$ is a unique solution to (2.10).

From the recurrence relations we can write

$$U_{n+1}(\underline{x}) - U_n(\underline{x}) = \int_{\underline{t} \varepsilon \omega(\underline{x})} \phi(\underline{x-t}) \, [\, U_n(\underline{t}) - U_{n-1}(\underline{t}) \,] \, \underline{dt} \qquad (n \geq 1)$$

Taking absolute value of both sides we obtain

$$|U_{n+1}(\underline{x}) - U_n(\underline{x})| \leq \int_{\underline{t} \varepsilon \omega(\underline{x})} \phi(\underline{x-t}) \, |U_n(\underline{t}) - U_{n-1}(\underline{t})| \, \underline{dt} \qquad (n \geq 1)$$

Let $m = |A(\underline{x})|$

Upon taking norms of both sides, of the above relations, we get for $n = 0, 1, 2, \ldots$

$$|U_1(\underline{x}) - U_0(\underline{x})| \leq m \int_{\underline{x-t} \varepsilon \omega(\underline{x})} \phi(\underline{t}) \, \underline{dt} = m\phi_{(1)}(\underline{x})$$

$$|U_2(\underline{x}) - U_1(\underline{x})| \le m\int_{\underline{x-t}\epsilon\omega(\underline{x})} \tilde{\Phi}_{(1)}\,\underline{(x-t)}\,\phi(\underline{t})\,\underline{dt} = m\phi_{(2)}(\underline{x})$$

$$\bullet$$
$$\bullet$$
$$\circ$$

and

$$|U_{n+1}(\underline{x}) - U_n(\underline{x})| \le m\int_{\underline{x-t}\epsilon\omega(\underline{x})} \tilde{\Phi}_{(n)}\,\underline{(x-t)}\,\phi(\underline{t})\,\underline{dt} = m\phi_{(n+1)}(\underline{x}) \qquad (n \ge 0)$$

With this inequality at our disposal we now show that the sequence of functions $\{U_n(\underline{x})\}$ is a Cauchy sequence with respect to the norm. The proof is as follows:

$$U_{n+p}(\underline{x}) - U_n(\underline{x}) = \sum_{j=0}^{p-1} [U_{n+j+1}(\underline{x}) - U_{n+j}(\underline{x})] \qquad (p > 1)$$

Taking norms of both sides, we obtain

$$\|U_{n+p}(\underline{x}) - U_n(\underline{x})\| \le \sum_{j=0}^{p-1} \|U_{n+j+1}(\underline{x}) - U_{n+j}(\underline{x})\| < m\sum_{j=0}^{p-1} \phi_{(n+j+1)}(\underline{x})$$

Since, from Remark 2.2, the series $\sum_{n=1}^{\infty} \phi_{(n)}(\underline{x})$ is uniformly convergent for every $\underline{x}\epsilon\omega(\underline{S})$ then the sequence of functions $\{U_n(\underline{x})\}$ converges to a function $\bar{U}(\underline{x})$ for every $\underline{x}\epsilon\omega(\underline{S})$. Moreover, $\bar{U}(\underline{x})$ is an element of A_1 since A_1 is a complete Banach space.

Let us find an analytical expression for $\bar{U}(\underline{x})$. Let $\hat{U}(\underline{x})$ be an initial solution for (2.10). Using the results from Lemma 2.1, Remark 2.3,

and the results of the g-convolution properties, we get by induction

for $N = 1, 2, \ldots$

$$U_1(\underline{x}) = A(\underline{x}) + \int_{\underline{t} \epsilon \omega(\underline{x})} \phi(\underline{x-t}) \, \hat{U}(\underline{t}) \, \underline{dt}$$

$$U_2(\underline{x}) = A(\underline{x}) + \int_{\underline{t} \epsilon \omega(\underline{x})} \phi(\underline{x-t}) \, [A(\underline{t}) + \int_{\underline{t}_1 \epsilon \omega(\underline{t})} \phi(\underline{t-t_1}) \, \hat{U}(\underline{t}_1) \, \underline{dt}_1] \, \underline{dt}$$

$$= A(\underline{x}) + \int_{\underline{t} \epsilon \omega(\underline{x})} \phi(\underline{x-t}) \, A(\underline{t}) \, \underline{dt}$$

$$+ \int_{\underline{t} \epsilon \omega(\underline{x})} \phi(\underline{x-t}) \, [\int_{\underline{t}_1 \epsilon \omega(\underline{x})} \phi(\underline{t-t_1}) \, \hat{U}(\underline{t}_1) \, \underline{dt}_1] \, \underline{dt}$$

$$= A(\underline{x}) + \int_{\underline{t} \epsilon \omega(\underline{x})} \phi(\underline{x-t}) \, A(\underline{t}) \, \underline{dt} + \int_{\underline{t} \epsilon \omega(\underline{x})} \phi_{(2)}(\underline{x-t}) \, \hat{U}(\underline{t}) \, \underline{dt}$$

and

$$U_N(\underline{x}) = A(\underline{x}) + \sum_{n=1}^{N-1} \int_{\underline{t} \epsilon \omega(\underline{x})} \phi_{(n)}(\underline{x-t}) \, A(\underline{t}) \, \underline{dt}$$

$$+ \int_{\underline{t} \epsilon \omega(\underline{x})} \phi_{(N)}(\underline{x-t}) \, \hat{U}(\underline{t}) \, \underline{dt} \qquad (N \geq 2)$$

where $\phi_{(n)}(\underline{x-t})$, $n \geq 1$, is the n-fold g-convolution of $\phi(\underline{x-t})$ with itself. Therefore, $\bar{U}(\underline{x})$ is given by

$$\bar{U}(\underline{x}) = A(\underline{x}) + \sum_{n=1}^{\infty} \int_{\underline{t} \epsilon \omega(\underline{x})} \phi_{(n)}(\underline{x-t}) \, A(\underline{t}) \, \underline{dt}$$

and (2.11) follows.

To show that (2.11) satisfies (2.10) we proceed as follows:
Using (2.11) we can write

$$U(\underline{x}) - \int_{\underline{t}\epsilon\omega(\underline{x})} \phi\underline{(x-t)}\ U(\underline{t})\ \underline{dt} = A(\underline{x}) + \int_{\underline{t}\epsilon\omega(\underline{x})} \psi\underline{(x-t)}\ A(\underline{t})\ \underline{dt}$$

$$- \int_{\underline{t}\epsilon\omega(\underline{x})} \phi\underline{(x-t)}\ [A(\underline{t}) + \int_{\underline{t_1}\epsilon\omega(\underline{t})} \psi\underline{(t-t_1)}\ A(\underline{t_1})\ \underline{dt_1}]\ \underline{dt}$$

Applying the g-convolution properties, Lemma 2.1, Remark 3.3, and the
definition of $\psi(\underline{t})$, we get

$$\overline{U}(\underline{x}) - \int_{\underline{t}\epsilon\omega(\underline{x})} \phi\underline{(x-t)}\ \overline{U}(\underline{t})\ \underline{dt} = A(\underline{x}) + \sum_{n=1}^{\infty} \int_{\underline{t}\epsilon\omega(\underline{x})} \phi_{(n)}\underline{(x-t)}\ A(\underline{t})\ \underline{dt}$$

$$- \int_{\underline{t}\epsilon\omega(\underline{x})} \phi\underline{(x-t)}\ A(\underline{t})\ \underline{dt} + \int_{\underline{t}\epsilon\omega(\underline{x})} \phi\underline{(x-t)} \cdot$$

$$[\sum_{n=1}^{\infty} \int_{\underline{t_1}\epsilon\omega(\underline{t})} \phi_{(n)}\underline{(t-t_1)}\ A(\underline{t_1})\ \underline{dt_1}]\ \underline{dt}$$

$$= A(\underline{x}) + \sum_{n=1}^{\infty} \int_{\underline{t}\epsilon\omega(\underline{x})} \phi_{(n)}\underline{(x-t)}\ A(\underline{t})\ \underline{dt} - \int_{\underline{t}\epsilon\omega(\underline{x})} \phi\underline{(x-t)}\ A(\underline{t})\ \underline{dt}$$

$$+ \sum_{n=1}^{\infty} \int_{\underline{t}\epsilon\omega(\underline{x})} \phi_{(n+1)}\underline{(x-t)}\ A(\underline{t})\ \underline{dt}$$

$$= A(\underline{x})$$

Thus (2.11) satisfies (2.10).

Conversely, we can show that if $U(\underline{x})$ satisfies (2.10), then
$U(\underline{x})$ satisfies (2.11). Using (2.10) we can write

$$A(\underline{x}) + \int_{\underline{t}\varepsilon\omega(\underline{x})} \psi(\underline{x-t})\ A(\underline{t})\ \underline{dt} = U(\underline{x}) - \int_{\underline{t}\varepsilon\omega(\underline{x})} \phi(\underline{x-t})\ U(\underline{t})\ \underline{dt}$$

$$+ \int_{\underline{t}\varepsilon\omega(\underline{x})} \psi(\underline{x-t})\ [\ U(\underline{t}) - \int_{\underline{t_1}\varepsilon\omega(\underline{t})} \phi(\underline{t-t_1})\ U(\underline{t_1})\ \underline{dt_1}]\ \underline{dt}$$

Using, again, Lemma 2.1, Remark 2.3, and the definition of $\psi(\underline{t})$, we get

$$A(\underline{x}) + \int_{\underline{t}\varepsilon\omega(\underline{x})} \psi(\underline{x-t})\ A(\underline{t})\ \underline{dt} = U(\underline{x}) - \int_{\underline{t}\varepsilon\omega(\underline{x})} \phi(\underline{x-t})\ U$$

$$+ \sum_{n=1}^{\infty} [\ \int_{\underline{t}\varepsilon\omega(\underline{x})} \phi_{(n)}(\underline{x-t})\ U(\underline{t})\ \underline{dt}\ ]$$

$$- \sum_{n=1}^{\infty} \int_{\underline{t}\varepsilon\omega(\underline{x})} \phi_{(n+1)}(\underline{x-t})\ U(\underline{t})\ \underline{dt}$$

$$= U(\underline{x})$$

Thus the solution for (2.10) is given by (2.11).

Now if the set Γ, as defined in Section 2.1, is admissible, then $\phi_{(n)}(\underline{t}) = \phi^{(n)}(\underline{t})$ where $\phi^{(n)}(\underline{t})$ is the n-fold, $n \geq 1$, ordinary convolution of $\phi(\underline{t})$ with itself. Thus the solution for (2.10) will be given by

$$U(\underline{x}) = A(\underline{x}) + \sum_{n=1}^{\infty} \int_{\underline{t}\varepsilon\omega(\underline{x})} \phi^{(n)}(\underline{x-t})\ A(\underline{t})\ \underline{dt}$$

Q. E. D.

2.4 Other Mathematical Concepts

As we shall see later the following theorem from calculus,
[23], will be needed in Chapters IV and V.

<u>Theorem 2.2</u>: Mean Value Theorem for multiple integrals.

If $f(\underline{t})$ is a continuous function on a compact set T for

which $A(T) = \int_T \underline{dt} \neq 0$, then there exists a point $\underline{v}$ in the interior

of T such that

$$\int_T f(\underline{t})\ \underline{dt} = f(\underline{v})\ A(T)$$

<u>Remark 2.4</u>:

From the theory of Linear Operators in Banach Space the
following results will be needed in Chapter III.

Define the operation

$$(Ta)\ (\underline{x}) = \int_{\underline{x-t}\varepsilon\omega(\underline{x})} a(\underline{x-t})\ \phi(\underline{t})\ \underline{dt} + \int_{\underline{x-t}\varepsilon\ r'(\underline{x})} a(\underline{x})\ \phi(\underline{t})\ \underline{dt}$$

$$\equiv \int_{\underline{x-t}\varepsilon r''(\underline{x})} a(\underline{x},\underline{t})\ \phi(\underline{t})\ \underline{dt}$$

$$\equiv \int_{\underline{v}\varepsilon r''(\underline{x})} \phi(\underline{x-v})\ a(\underline{x},\underline{v})\ \underline{dv} \qquad \underline{x}\varepsilon\omega(\underline{S})$$

With $\|T^n\| = 1$, T^n is the nth iterate of T, for $n = 1, 2, \ldots$
T is defined for all functions $a(\underline{v})$ which are bounded for all $\underline{v}\varepsilon r''(\underline{x})$,
$\underline{x}\varepsilon\omega(\underline{S})$. The collection of all such functions forms a Banach (M) Space
in which the norm of an element is defined to be

$$\sup_{\underline{v} \in r''(\underline{S})} |a(\underline{v})|$$

For such integral operation T with bounded density kernel, Yosida and Kakutan [22] showed that

$$T^n = \sum_{i=1}^{k} T_i + P^n$$

$$= B + P^n \qquad n = 1, 2, \ldots .$$

where B is the ergodic part of the operation T^n and P^n is the non-ergodic part of T^n, k is the number of proper values of T, T_i (i = 1, 2, . . ., k) is a completely continuous linear operation and P^n (n $\geq$ 1) is a completely continuous linear operation (which might vanish) with

$$\|P^n\| \leq \frac{r}{(1+e)^n}$$

where r and e are positive constants independent of n.

CHAPTER III

MATHEMATICAL FORMULATION

3.1 Introduction

An m-commodity ($m \geq 1$) inventory control system operating
under a (σ, S) policy is considered. At the beginning of each period
a decision to order or not to order is made depending on the stock
level $\underline{x} = (x_1, x_2, \ldots, x_m)$. The ordering decision in each period
is affected, as discussed previously in Chapter I, by a single fixed
set-up cost K, a linear variable ordering cost $\underline{c} = (c_1, c_2, \ldots, c_m)$,
and the expected holding and shortage cost $L(\underline{x})$ conditional on being
in stock level $\underline{x}$ at the beginning of a period. Demand, $\{\underline{D}_i\}$, for the
items over a sequence of periods ($i = 1, 2, \ldots$), is assumed to be
independently and identically distributed continuous random variables
with joint density function $\phi_{\underline{D}}(\underline{t})$, $\underline{t} \geq \underline{0}$, $\underline{t} = (t_1, t_2, \ldots, t_m)$.
Immediate delivery of orders and complete backlogging of unfilled demands
are assumed for all commodities.

As discussed in Chapter I, under the adopted stationary (σ, S)
policy, the sequence of stock levels at the beginning of each period
forms a discrete-time Markov process.

This chapter is devoted to finding an expression for the
stationary total expected cost per period based on the assumed ordering
policy.

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3.2 Mathematical Formulation

For an n period problem $(n \geq 1)$, let

$\underline{x}$ = initial stock level for the items prior to making a decision.

$f_n(\underline{x})$ = total expected cost of operating the system for n periods when no order is placed initially, i.e., $\underline{x}\epsilon\sigma^C$.

$h_n(\underline{x})$ = total expected cost of operating the system for n periods when an order is placed initially, i.e., $\underline{x}\epsilon\sigma$.

If no order is placed initially, i.e., $\underline{x}\epsilon\sigma^C$, then $f_n(\underline{x})$ satisfies the functional equation

$$f_1(\underline{x}) = L(\underline{x}) \qquad (n = 1)$$

$$f_n(\underline{x}) = L(\underline{x}) + \int_{\underline{x}-t\epsilon\omega(\underline{x})} f_{n-1}(\underline{x-t})\ \phi(\underline{t})\ \underline{dt}$$

$$+ \int_{\underline{x}-t\epsilon r^1(\underline{x})} h_{n-1}(\underline{x-t})\ \phi(\underline{t})\ \underline{dt} \qquad (n \geq 2) \qquad (3.1)$$

However, if an order is placed initially, i.e., $\underline{x}\epsilon\sigma$, then $h_n(\underline{x})$ satisfies the functional equation

$$h_1(x) = K + c^T(\underline{S-x}) + L(\underline{S}) \qquad\qquad (n = 1)$$

$$h_n(\underline{x}) = K + c^T(\underline{S-x}) + L(\underline{S})$$

$$+ \int_{\underline{S-t\epsilon\omega(\underline{S})}} f_{n-1}(\underline{S-t})\ \phi(\underline{t})\ d\underline{t} \qquad\qquad\qquad (3.2)$$

$$+ \int_{\underline{S-t\epsilon r^1(\underline{S})\Xi\sigma}} h_{n-1}(\underline{S-t})\ \phi(\underline{t})\ d\underline{t} \qquad (n \geq 2)$$

From (3.1) and (3.2)

$$h_n(\underline{x}) = K + c^T(\underline{S-x}) + f_n(\underline{S}) \qquad\qquad \underline{x}\epsilon\sigma\ (n \geq 1) \qquad (3.3)$$

To find, explicitly, an analytic expression for the stationary total expected cost per period, the following theorem, based on Howard [6] and White [2] work, is needed:

Theorem 3.1

For large n

(a) $h_n(\underline{x}) = ng + v(\underline{x})$ $\underline{x}\epsilon\sigma$

(b) $f_n(\underline{x}) = ng + u(\underline{x})$ $\underline{x}\epsilon\sigma^c$

where g is the stationary total expected cost of the process per period and $u(\underline{x})$ and $v(\underline{x})$ are functions of the initial state of the process.

Proof: From (3.1) and (3.3) for $\underline{x}\epsilon\sigma^c$

$$f_1(\underline{x}) = L(\underline{x})$$

$$f_n(\underline{x}) = L(\underline{x}) + \int_{\underline{x-t\epsilon\omega(\underline{x})}} f_{n-1}(\underline{x-t})\ \phi(\underline{t})\ d\underline{t}$$

$$+ \int_{\underline{x-t\epsilon r^1(\underline{x})}} \{K + c^T[\underline{S-(x-t)}]\}\ \phi(\underline{t})\ d\underline{t} \qquad\qquad (3.4)$$

$$+ f_{n-1}(\underline{S}) \int_{\underline{x-t\epsilon r^1(\underline{x})}} \phi(\underline{t})\ d\underline{t} \qquad\qquad (n \geq 2)$$

Let us denote by

$$a(\underline{x}) = \int_{\underline{x-t}\epsilon r^1(\underline{x})} \{K + c^T(\underline{S}-(\underline{x-t}))\}\, \phi(\underline{t})\, \underline{dt} \qquad (3.5)$$

$$b_{n-1}(\underline{S},\underline{x}) = f_{n-1}(\underline{S}) \int_{\underline{x-t}\epsilon r^1(\underline{x})} \phi(\underline{t})\, \underline{dt} \qquad (n \geq 2) \qquad (3.6)$$

Upon using (3.5) and (3.6) in (3.4) we get

$$f_1(\underline{x}) = L(\underline{x})$$

$$f_n(\underline{x}) = L(\underline{x}) + a(\underline{x}) + b_{n-1}(\underline{S},\underline{x}) + \int_{\underline{t}\epsilon\omega(\underline{x})} \phi(\underline{x-t})\, f_{n-1}(\underline{t})\, \underline{dt} \qquad (n \geq 2)$$

From the above relation on using the g-convolution properties and Remark 2.3, we obtain by induction for $n = 2, 3, \ldots$

$$f_2(\underline{x}) = L(\underline{x}) + a(\underline{x}) + b_1(\underline{S},\underline{x}) + \int_{\underline{t}\epsilon\omega(\underline{x})} \phi(\underline{x-t})\, L(\underline{t})\, \underline{dt}$$

$$f_3(\underline{x}) = L(\underline{x}) + a(\underline{x}) + b_2(\underline{S},\underline{x}) + \int_{\underline{t}\epsilon\omega(\underline{x})} \phi(\underline{x-t}) \cdot$$

$$[L(\underline{t}) + a(\underline{t}) + b_1(\underline{S},\underline{t}) + \int_{\underline{t_1}\epsilon\omega(\underline{t})} \phi(\underline{t-t_1})\, L(\underline{t_1})\, \underline{dt_1}]\, \underline{dt}$$

$$= L(\underline{x}) + a(\underline{x}) + b_2(\underline{S},\underline{x}) + \int_{\underline{t}\epsilon\omega(\underline{x})} \phi(\underline{x-t})\, [L(\underline{t}) + a(\underline{t})$$

$$+ b_1(\underline{S},\underline{t})]\, \underline{dt} + \int_{\underline{t}\epsilon\omega(\underline{x})} \phi(\underline{x-t}) \cdot$$

$$[\int_{\underline{t_1}\epsilon\omega(\underline{t})} \phi(\underline{t-t_1})\, L(\underline{t_1})\, \underline{dt_1}]\, \underline{dt}$$

$$= L(\underline{x}) + a(\underline{x}) + b_2(\underline{S},\underline{x}) + \int_{\underline{t}\in\omega(\underline{x})} \phi(\underline{x-t}) \, [L(\underline{t}) + a(\underline{t})$$

$$+ b_1(\underline{S},\underline{t})] \, \underline{dt} + \int_{\underline{t}\in\omega(\underline{x})} \phi_{(2)}(\underline{x-t}) \, L(\underline{t}) \, \underline{dt}$$

.
.
.

and

$$f_n(\underline{x}) = L(\underline{x}) + a(\underline{x}) + b_{n-1}(\underline{S},\underline{x}) + \sum_{i=1}^{n-2} \int_{\underline{t}\in\omega(\underline{x})} \phi_{(i)}(\underline{x-t}) \cdot$$

$$[L(\underline{t}) + a(\underline{t}) + b_{n-2+1-i}(\underline{S},\underline{t})] \, \underline{dt}$$

$$+ \int_{\underline{t}\in\omega(\underline{x})} \phi_{(n-1)}(\underline{x-t}) \, L(\underline{t}) \, \underline{dt} \qquad (n > 2)$$

where $\phi_{(n)}(\underline{x-t})$, $n \geq 1$, is the n-fold g-convolution of $\phi(\underline{x-t})$ with itself. On regrouping terms, we get

$$f_n(\underline{x}) = [L(\underline{x}) + \sum_{i=1}^{n-1} \int_{\underline{t}\in\omega(\underline{x})} \phi_i(\underline{x-t}) \, L(\underline{t}) \, \underline{dt} \,]$$

$$+ [a(\underline{x}) + \sum_{i=1}^{n-2} \int_{\underline{t}\in\omega(\underline{x})} \phi_{(i)}(\underline{x-t}) \, a(\underline{t}) \, \underline{dt}]$$

$$+ [b_{n-1}(\underline{S},\underline{x}) + \sum_{i=1}^{n-2} \int_{\underline{t}\in\omega(\underline{x})} \phi_{(i)}(\underline{x-t}) \, b_{n-2+1-i}(\underline{S},\underline{t}) \, \underline{dt}]$$

$$(n \geq 2) \qquad (3.7)$$

To proceed in the proof, first, we shall simplify (3.7) by substituting for $a(\underline{x})$ as given by (3.5). Next, we shall find an analytical expression for $f_n(\underline{S})$ and use it in finding the expression for $f_n(\underline{x})$, $\underline{x}\varepsilon\sigma^C$, for large n. Finally, we shall determine the expression for $h_n(\underline{x})$, $\underline{x}\varepsilon\sigma$, for large values of n.

Let $\underline{\mu} = (\mu_1, \mu_2, \ldots, \mu_m)$, where μ_i is the expected value of the random variable $\underline{D}_i$, $i = 1, 2, \ldots, m$. From (3.5) and the definition of the sets $\omega(\underline{x})$ and $r'(\underline{x})$, we have

$$a(\underline{x}) = \int_{\underline{x-t}\varepsilon r'(\underline{x})} [K + c^T(\underline{S}-(\underline{x-t}))] \, \phi(\underline{t}) \, \underline{dt}$$

$$= \int_{\underline{0}}^{\infty} [K + c^T(\underline{S}-(\underline{x-t}))] \, \phi(\underline{t}) \, \underline{dt}$$

$$- \int_{\underline{t}\varepsilon\omega(\underline{x})} [K + c^T(\underline{S-t})] \, \phi(\underline{x-t}) \, \underline{dt}$$

$$= K + c^T(\underline{S}-(\underline{x-\mu})) - \int_{\underline{t}\varepsilon\omega(\underline{x})} [K + c^T(\underline{S-t})] \, \phi(\underline{x-t}) \, \underline{dt}$$

Upon using the g-convolution properties and Lemma 2.1 we obtain for $i = 1, 2, \ldots$

$$\int_{\underline{t}\varepsilon\omega(\underline{x})} \phi_{(1)}(\underline{x-t}) \, a(\underline{t}) \, \underline{dt} = \int_{\underline{t}\varepsilon\omega(\underline{x})} \phi_{(1)}(\underline{x-t}) \, \{K + c^T(\underline{S}-(\underline{t-\mu}))$$

$$- \int_{\underline{t_1}\varepsilon\omega(\underline{t})} [K + c^T(\underline{S-t_1})] \, \phi(\underline{t-t_1}) \, \underline{dt_1}\} \, \underline{dt}$$

$$= \int_{\underline{t}\varepsilon\omega(\underline{x})} \phi\,(\underline{x-t})\ [K + c^T(\underline{S} - (\underline{t-\mu}))]\ \underline{dt}$$

$$- \int_{\underline{t}\varepsilon\omega(\underline{x})} \phi_{(2)}\,(\underline{x-t})\ [K + c^T(\underline{S-t})]\ \underline{dt}$$

and,

$$\int_{\underline{t}\varepsilon\omega(\underline{x})} \phi_{(i)}\,(\underline{x-t})\ a(\underline{t})\ \underline{dt} = \int_{\underline{t}\varepsilon\omega(\underline{x})} \phi_{(i)}\,(\underline{x-t})\ [K + c^T(\underline{S} - (\underline{t-\mu}))]\ \underline{dt}$$

$$- \int_{\underline{t}\varepsilon\omega(\underline{x})} \phi_{(i+1)}\,(\underline{x-t})\ [K + c^T(\underline{S-t})]\ \underline{dt} \qquad (i \geq 1)$$

Upon using the above relations, we obtain

$$a(\underline{x}) + \sum_{i=1}^{n-2} \int_{\underline{t}\varepsilon\omega(\underline{x})} \phi_{(i)}\,(\underline{x-t})\ a(\underline{t})\ \underline{dt} = K + c^T(\underline{S-x}) + c^T\underline{\mu}$$

$$- \int_{\underline{t}\varepsilon\omega(\underline{x})} \{K + c^T(\underline{S-t})\}\ \phi\,(\underline{x-t})\ \underline{dt}$$

$$+ \sum_{i=1}^{n-2} \left[\int_{\underline{t}\varepsilon\omega(\underline{x})} \phi_{(i)}\,(\underline{x-t})\ [K + c^T(\underline{S-t+\mu})]\ \underline{dt} \right.$$

$$\left. - [\ \int_{\underline{t}\varepsilon\omega(\underline{x})} \phi_{(i+1)}\,(\underline{x-t})\ [K + c^T(\underline{S-t})]\ \underline{dt} \right]$$

On simplifying and regrouping terms we get

$$a(\underline{x}) + \sum_{i=1}^{n-2} \int_{\underline{t}\varepsilon\omega(\underline{x})} \phi_{(i)}\,(\underline{x-t})\ a(\underline{t})\ \underline{dt}$$

$$= K + c^T\underline{\mu}\ [1 + \sum_{i=1}^{n-2} \int_{\underline{t}\varepsilon\omega(\underline{x})} \phi_{(i)}\,(\underline{x-t})\ \underline{dt}] + c^T(\underline{S-x})$$

$$- \int_{\underline{t} \epsilon \omega(\underline{x})} \phi(n-1) \, \underline{(x-t)} \, [K + c^T(\underline{S-t})] \, \underline{dt} \tag{3.8}$$

Now, we shall proceed to find an analytical expression for $f_n(\underline{S})$, $n \geq 2$.

From (3.4), for $\underline{x} = \underline{S}$, we have

$$
\left.
\begin{aligned}
f_1(\underline{S}) &= L(\underline{S}) \\
f_n(\underline{S}) &= L(\underline{S}) + \int_{\underline{S-t} \epsilon \omega(\underline{S})} f_{n-1} \underline{(S-t)} \, \phi(\underline{t}) \, \underline{dt} \\
&\quad + \int_{\underline{S-t} \epsilon r^1(\underline{S})} [K + c^T \underline{t}] \, \phi(\underline{t}) \, \underline{dt} \\
&\quad + f_{n-1}(\underline{S}) \int_{\underline{S-t} \epsilon r^1(\underline{S})} \phi(\underline{t}) \, \underline{dt} \qquad (n \geq 2)
\end{aligned}
\right\} \tag{3.9}
$$

Let us denote by

$$c(\underline{S}) = \int_{\underline{S-t} \epsilon r'(\underline{S})} [K + c^T \underline{t}] \, \phi(\underline{t}) \, \underline{dt} \tag{3.10}$$

$$
\begin{aligned}
T(f_{n-1}(\underline{t}))(\underline{S}) &= \int_{\underline{S-t} \epsilon \omega(\underline{S})} f_{n-1} \underline{(S-t)} \, \phi(\underline{t}) \, \underline{dt} \\
&\quad + f_{n-1}(\underline{S}) \int_{\underline{S-t} \epsilon r'(\underline{S})} \phi(\underline{t}) \, \underline{dt} \qquad (n \geq 2)
\end{aligned}
\tag{3.11}
$$

Note T as defined is a bounded linear operation. Substituting (3.10) and (3.11) in (3.9) we get

$$
\left.
\begin{aligned}
f_1(\underline{S}) &= L(\underline{S}) \\
f_n(\underline{S}) &= L(\underline{S}) + T(f_{n-1}(t))(\underline{S}) + c(\underline{S}) \qquad (n \geq 2)
\end{aligned}
\right\} \tag{3.12}
$$

From (3.12), by induction for n = 2, 3, . . ., we have

$$f_2(\underline{S}) = L(\underline{S}) + c(\underline{S}) + T(L(\underline{t}))(\underline{S})$$

$$f_3(\underline{S}) = L(\underline{S}) + c(\underline{S}) + T(L(\underline{t}) + c(\underline{t}) + T(L(\underline{t})))(\underline{S})$$

. . . .

. . . .

. . . .

and,

$$f_n(\underline{S}) = L(\underline{S}) + c(\underline{S}) + \sum_{i=1}^{n-2} T^i(L(\underline{t}) + c(\underline{t}))(\underline{S})$$

$$+ T^{n-1}(L(\underline{t}))(\underline{S}) \qquad (n \geq 2) \qquad (3.13)$$

From Yosida and Kakutan [22] we have

$$T^i = \sum_{j=1}^{k} T_j + P^i$$

$$= B + P^i \qquad (i = 1, 2, \ldots) \qquad (3.14)$$

where B is independent of i. The linear operations T_j (j = 1, 2, . . . , k) and P^i are explicitly defined in Remark 2.4. On substituting (3.14) in (3.13), we get

$$f_n(\underline{S}) = L(\underline{S}) + c(\underline{S}) + [.(B+P^1) + (B+P^2) + \ldots$$

$$+ (B+P^{n-2})] \; L(\underline{t}) + c(\underline{t}))(\underline{S})$$

$$+ [B+P^{n-1}] \; (L(\underline{t}))(\underline{S}) \qquad (n \geq 2)$$

Upon regrouping terms, we obtain

$$f_n(\underline{S}) = L(\underline{S}) + c(\underline{S}) + (n-1) \ B(L(\underline{t}))(\underline{S})$$

$$+ (n-2) \ B(c(\underline{t}))(\underline{S}) + [\sum_{i=1}^{n-1} P^i] \ (L(\underline{t}))(\underline{S})$$

$$+ [\sum_{i=1}^{n-2} P^i] \ c(\underline{t})(\underline{S}) \qquad (n \geq 2) \qquad (3.15)$$

From (3.6) and the definition of the sets $\omega(\underline{S})$ and $r^1(\underline{S})$, we can write

$$b_{n-1}(\underline{S},\underline{x}) = f_{n-1}(\underline{S}) \int_{\underline{x-t} \in r^1(\underline{x})} \phi(\underline{t}) \ \underline{dt}$$

$$= f_{n-1}(\underline{S}) \ [\int_{\underline{0}}^{\infty} \phi(\underline{t}) \ \underline{dt} - \int_{\underline{t} \in \omega(\underline{x})} \phi(\underline{x-t}) \ \underline{dt}]$$

$$= f_{n-1}(\underline{S}) \ [1 - \int_{\underline{t} \in \omega(\underline{x})} \phi(\underline{x-t}) \ \underline{dt}] \qquad (n \geq 2)$$

and for $n \geq 2$

$$b_{n-1}(\underline{S},\underline{x}) + \sum_{i=1}^{n-2} \int_{\underline{t} \in \omega(\underline{x})} \phi_{(i)}(\underline{x-t}) \ b_{n-1-i}(\underline{S},\underline{t}) \ \underline{dt}$$

$$= f_{n-1}(\underline{S}) \ [1 - \int_{\underline{t} \in \omega(\underline{x})} \phi(\underline{x-t}) \ \underline{dt}] + [\sum_{i=1}^{n-2} f_{n-1-i}(\underline{S}) \cdot$$

$$\int_{\underline{t} \in \omega(\underline{x})} \phi_{(i)}(\underline{x-t}) \ [1 - \int_{\underline{t_1} \in \omega(\underline{t})} \phi(\underline{t-t_1}) \ \underline{dt_1}] \ \underline{dt}]$$

On applying the g-convolution properties and Lemma 2.1, we can write

$$b_{n-1}(\underline{S},\underline{x}) + \sum_{i=1}^{n-2} \int_{\underline{t} \in \omega(\underline{x})} \phi_{(i)}(\underline{x-t}) \ b_{n-1-i}(\underline{S},\underline{t}) \ \underline{dt}$$

$$= f_{n-1}(\underline{S}) \ [1 - \int_{\underline{t}\epsilon\omega(\underline{x})} \phi\underline{(x-t)} \ \underline{dt}] + \sum_{i=1}^{n-2} f_{n-1-i}(\underline{S}) \ \cdot$$

$$[\int_{\underline{t}\epsilon\omega(\underline{x})} \phi_{(i)} \underline{(x-t)} \ \underline{dt} - \int_{\underline{t}\epsilon\omega(\underline{x})} \phi_{(i+1)} \underline{(x-t)} \ \underline{dt} \ ] \ (n \geq 2) \tag{3.16}$$

Denote by $F^i = \int_{\underline{t}\epsilon\omega(\underline{x})} \phi_{(i)} \underline{(x-t)} \ \underline{dt}$ $\qquad$ $(i = 1, 2, \ldots .)$ $\tag{3.17}$

with $F^0 = 1.$

Using (3.17) in (3.16) and simplifying, we get

$$b_{n-1}(\underline{S},\underline{x}) + \sum_{i=1}^{n-2} \int_{\underline{t}\epsilon\omega(\underline{x})} \phi_{(i)} \underline{(x-t)} \ b_{n-1-i}(\underline{S},\underline{t}) \ \underline{dt}$$

$$= \sum_{i=1}^{n-1} f_{n-i}(\underline{S}) \ [F^{i-1} - F^i]$$

$$= \sum_{i=1}^{n-3} f_{n-1}(\underline{S}) \ [F^{i-1} - F^i] + f_2(\underline{S}) \ [F^{n-3} - F^{n-2}]$$

$$+ f_1(\underline{S}) \ [F^{n-2} - F^{n-1}] \tag{3.18}$$

On substituting (3.15) in (3.18) and expanding we can write for $n \geq 2$

$$b_{n-1}(\underline{S},\underline{x}) + \sum_{i=1}^{n-2} \int_{\underline{t}\epsilon\omega(\underline{x})} \phi_{(i)} \underline{(x-t)} \ b_{n-1-i}(\underline{S},\underline{t}) \ \underline{dt}$$

$$= \sum_{i=1}^{n-3} \ [L(\underline{S}) + c(\underline{S}) + (n-1-i) \ B(L(\underline{t}))(\underline{S})$$

$$+ (n-2-i) \ B(c(\underline{t}))(\underline{S}) + [\sum_{j=1}^{n-1-i} P^j] \ (L(\underline{t}))(\underline{S})$$

$$+ [\sum_{j=1}^{n-2-i} P^j] \ (c(\underline{t}))(\underline{S})] \ [F^{i-1} - F^i]$$

$$+ L(\underline{S}) + c(\underline{S}) + B(L(\underline{t}))(\underline{S}) + P^1(L(\underline{t}))(\underline{S})] \ [F^{n-3} - F^{n-2}]$$

$$+ L(\underline{S}) \ [F^{n-2} - F^{n-1}]$$

$$= [L(\underline{S}) + c(\underline{S})] \ [F^0 - F^1 + F^1 - F^2 + \ldots + F^{n-4} - F^{n-3}$$

$$+ F^{n-3} - F^{n-2}] + L(\underline{S}) \ [F^{n-2} - F^{n-1}] + B(L(\underline{t}))(\underline{S})$$

$$\left[(n-2) \ [F^0 - F^1] + (n-3) \ [F^1 - F^2] + \ldots \right.$$

$$\left. + 2[F^{n-4} - F^{n-3}] + [F^{n-3} - F^{n-2}] \right] + B(c(\underline{t}))(\underline{S})$$

$$\left[(n-3) \ [F^0 - F^1] + (n-4) \ [F^1 - F^2] + \ldots + \right.$$

$$\left. [F^{n-4} - F^{n-3}] \right] + [\ \sum_{j=1}^{n-2} P^j] \ (L(\underline{t}))(\underline{S}) \ [F^0 - F^1]$$

$$+ [\ \sum_{j=1}^{n-3} P^j] \ (L(\underline{t}))(\underline{S}) \ [F^1 - F^2] + \ldots + [\ \sum_{j=1}^{2} P^j]$$

$$(L(\underline{t}))(\underline{S}) \ [F^{n-4} - F^{n-3}] + P^1(L(\underline{t}))(\underline{S}) \ [F^{n-3} - F^{n-2}]$$

$$+ [\ \sum_{j=1}^{n-3} P^j] \ (c(\underline{t}))(\underline{S}) \ [F^0 - F^1] + [\ \sum_{j=1}^{n-4} P^j] \ (c(\underline{t}))(\underline{S})$$

$$[F^1 - F^2] + \ldots + P^1(c(\underline{t}))(\underline{S}) \ [F^{n-4} - F^{n-3}]$$

On simplifying, we obtain

$$b_{n-1}(\underline{S},\underline{x}) + \sum_{i=1}^{n-2} \int_{\underline{t}\varepsilon\omega(\underline{x})} \phi(i) (\underline{x-t}) \, b_{n-1-i}(\underline{S},\underline{t}) \, \underline{dt}$$

$$= L(\underline{S}) + c(\underline{S}) - c(\underline{S}) F^{n-2} - L(\underline{S})F^{n-1} + (n-2) B(L(\underline{t}))(\underline{S})$$

$$- B(L(\underline{t}))(\underline{S}) \sum_{i=1}^{n-2} F^i + (n-3) B(c(\underline{t}))(\underline{S})$$

$$- B(c(\underline{t}))(\underline{S}) \sum_{i=1}^{n-3} F^i + [\sum_{j=1}^{n-2} P^j] (L(\underline{t}))(\underline{S})$$

$$+ [-P^{n-2}(L(\underline{t}))(\underline{S})F^1 - P^{n-3}(L(\underline{t}))F^2 - \ldots$$

$$- P^1(L(\underline{t}))(\underline{S})F^{n-2}] + [-P^{n-3}(c(\underline{t}))(\underline{S})F^1 - P^{n-4}(c(\underline{t})(\underline{S})F^2$$

$$- \ldots - P^1(c(\underline{t}))(\underline{S})F^{n-3}] + [\sum_{j=1}^{n-3} P^j] (c(\underline{t}))(\underline{S})$$

$$= L(\underline{S}) + c(\underline{S}) - c(\underline{S})F^{n-2} - L(\underline{S})F^{n-1} + (n-2) B(L(\underline{t}))(\underline{S})$$

$$- B(L(\underline{t}))(\underline{S}) \sum_{i=1}^{n-2} F^i + (n-3) B(c(\underline{t}))(\underline{S})$$

$$- B(c(\underline{t}))(\underline{S}) \sum_{i=1}^{n-3} F^i + [\sum_{i=1}^{n-2} P^i] (L(\underline{t}))(\overline{\underline{S}})$$

$$+ [\sum_{i=1}^{n-3} P^i] (c(\underline{t}))(\underline{S}) - \sum_{i=1}^{n-2} P^{n-1-i}(L(\underline{t}))(\underline{S})F^i$$

$$- \sum_{i=1}^{n-3} P^{n-2-i} (c(\underline{t}))(\underline{S})F^i \qquad (3.19)$$

Let

$$D_{n-2}(\underline{S},\underline{x}) = [\sum_{i=1}^{n-2} P^i] \, (L(\underline{t}))(\underline{S}) + [\sum_{i=1}^{n-3} P^i] \, (c(\underline{t}))(\underline{S})$$

$$- \sum_{i=1}^{n-2} P^{n-1-i}(L(\underline{t}))(\underline{S})F^i$$

$$- \sum_{i=1}^{n-3} P^{n-2-i}(c(\underline{t}))(\underline{S})F^i \qquad (n \geq 2) \qquad (3.20)$$

Then (3.19) becomes (after substituting for F^i, $i \geq 1$)

$$b_{n-1}(\underline{S},\underline{x}) + \sum_{i=1}^{n-2} \int_{\underline{t}\epsilon\omega(\underline{x})} \phi_{(i)}\underline{(x-t)} \, b_{n-1-i}(\underline{S},\underline{t}) \, \underline{dt}$$

$$= L(\underline{S}) + c(\underline{S}) - c(\underline{S}) \int_{\underline{t}\epsilon\omega(\underline{x})} \phi_{(n-2)}\underline{(x-t)} \, \underline{dt}$$

$$- L(\underline{S}) \int_{\underline{t}\epsilon\omega(\underline{x})} \phi_{(n-1)}\underline{(x-t)} \, \underline{dt} + (n-2) \, B(L(\underline{t}))(\underline{S})$$

$$+ (n-3) \, B(c(\underline{t}))(\underline{S}) - B(L(\underline{t}))(\underline{S}) \sum_{i=1}^{n-2} \int_{\underline{t}\epsilon\omega(\underline{x})} \phi_{(i)} \underline{(x-t)} \, \underline{dt}$$

$$- B(c(\underline{t}))(\underline{S}) \sum_{i=1}^{n-3} \int_{\underline{t}\epsilon\omega(\underline{x})} \phi_{(i)} \underline{(x-t)} \, \underline{dt} + D_{n-2}(\underline{S},\underline{x})$$

$$(n \geq 2) \qquad (3.21)$$

Using (3.8) and (3.21) in (3.7) we get

$$f_n(\underline{x}) = [L(\underline{x}) + \sum_{i=1}^{n-1} \int_{\underline{t}\epsilon\omega(\underline{x})} \phi_{(i)} \underline{(x-t)} \, L(\underline{t}) \, \underline{dt}]$$

$$+ [K + c^T \underline{\mu} \ [1 + \sum_{i=1}^{n-2} \int_{\underline{t} \epsilon \omega(\underline{x})} \phi_{(i)} \underline{(x-t)} \ \underline{dt} \ ]$$

$$+ c^T \underline{(S-x)} - \int_{\underline{t} \epsilon \omega(\underline{x})} \phi_{(n-1)} \underline{(x-t)} \ [K + c^T \underline{(S-t)}] \ \underline{dt}]$$

$$+ L(\underline{S}) + c(\underline{S}) - c(\underline{S}) \int_{\underline{t} \epsilon \omega(\underline{x})} \phi_{(n-2)} \underline{(x-t)} \ \underline{dt}$$

$$- L(\underline{S}) \int_{\underline{t} \epsilon \omega(\underline{x})} \phi_{(n-1)} \underline{(x-t)} \ \underline{dt} + (n-2) \ B(L(\underline{t}))(\underline{S})$$

$$+ (n-3) \ B(c(\underline{t}))(\underline{S}) - B(L(\underline{t}))(\underline{S}) \sum_{i=1}^{n-2} \int_{\underline{t} \epsilon \omega(\underline{x})} \phi_{(i)} \underline{(x-t)} \ \underline{dt}$$

$$- B(c(\underline{t}))(\underline{S}) \sum_{i=1}^{n-3} \int_{\underline{t} \epsilon \omega(\underline{x})} \phi_{(i)} \underline{(x-t)} \ \underline{dt} + D_{n-2} \underline{(S,x)}$$

$$(n \geq 2) \qquad (3.22)$$

Next, we shall study the asymptotic behavior of $f_n(\underline{x})$ for large n. For

n large we have, because of the convergence of $\sum_{i=1}^{\infty} \phi_{(i)} \underline{(x-t)}$ to $\psi \underline{(x-t)}$,

$$\phi_{(n)} \underline{(x-t)} \to 0$$

$$\sum_{i=1}^{n} \phi_{(i)} \underline{(x-t)} \to \psi \underline{(x-t)}$$

Moreover, from Remark 2.4, P^i, $i \geq 1$, is a completely continuous linear
operation (which might vanish) with

$$\|P^i\| \leq \frac{r}{(1+e)^i} < 1 \qquad i = 1, 2, \ldots$$

where r and e are positive constants independent of i. Thus, for large n, $D_{n-2}(\underline{S},\underline{x})$ as given by (3.20) converges to a function $D(\underline{S},\underline{x})$ which decreases rapidly as n becomes large. Using these last remarks in (3.22) we get, for large n,

$$f_n(\underline{x}) = L(\underline{x}) + \int_{\underline{t}\epsilon\omega(\underline{x})} \psi(\underline{x-t})\ L(\underline{t})\ \underline{dt} + K$$

$$+ c^T\underline{\mu}[1 + \int_{\underline{t}\epsilon\omega(\underline{x})} \psi(\underline{x-t})\ \underline{dt}] + c^T(\underline{S-x}) + L(\underline{S})$$

$$+ c(\underline{S}) + nB(L(\underline{t})+c(\underline{t}))(\underline{S}) - 2B(L(\underline{t}))(\underline{S}) - 3B(c(\underline{t}))(\underline{S})$$

$$- B(L(\underline{t}))(\underline{S}) \int_{\underline{t}\epsilon\omega(\underline{x})} \psi(\underline{x-t})\ \underline{dt}$$

$$- B(c(\underline{t}))(\underline{S}) \int_{\underline{t}\epsilon\omega(\underline{x})} \psi(\underline{x-t})\ \underline{dt} + D(\underline{S},\underline{x})$$

Or,

$$f_n(\underline{x}) = ng + u(\underline{x}) \qquad\qquad \underline{x}\epsilon\sigma^c \qquad\qquad (3.23)$$

where $g = B(L(\underline{t})+c(\underline{t}))(\underline{S})$ is independent of n, and $u(\underline{x})$ is a function of $\underline{x}$.

Now from (3.3) for large n

$$h_n(\underline{x}) = K + c^T(\underline{S-x}) + f_n(\underline{S})$$

$$= K + c^T(\underline{S-x}) + ng + u(\underline{S})$$

$$= ng + v(\underline{x}) \qquad\qquad \underline{x}\epsilon\sigma \qquad\qquad (3.24)$$

where,

$$v(\underline{x}) = K + c^T(\underline{S}-\underline{x}) + \dot{u}(\underline{S}) \qquad \underline{x}\epsilon\sigma. \qquad (3.25)$$

Q. E. D.

Theorem 3.2:

The stationary total expected cost per period, g, is

$$g = \frac{K + L(\underline{S}) + \displaystyle\int_{\underline{S}-\underline{t}\epsilon\omega(S)} L(\underline{S}-\underline{t})\ \psi(\underline{t})\ \underline{dt}}{1 + \displaystyle\int_{\underline{S}-\underline{t}\epsilon\omega(S)} \psi(\underline{t})\ \underline{dt}} + c^T\underline{\mu} \qquad (3.26)$$

where $\psi(\underline{t}) = \displaystyle\sum_{n=1}^{\infty} \phi_{(n)}(\underline{t})$ and $\underline{\mu}$ is the expectation vector of the

random variable $\underline{D}$ with joint density function $\phi_{\underline{D}}(\underline{t})$.

Proof: For large value of n, using (3.23) and (3.24) in relation to (3.1), we obtain

$$ng + u(\underline{x}) = L(\underline{x}) + \int_{\underline{x}-\underline{t}\epsilon\omega(x)} \big((n-1)g + u(\underline{x}-\underline{t})\big)\ \phi(\underline{t})\ \underline{dt}$$

$$+ \int_{\underline{x}-\underline{t}\epsilon r'(\underline{x})} \big((n-1)g + v(\underline{x}-\underline{t})\big)\ \phi(\underline{t})\ \underline{dt} \qquad (3.27)$$

From (3.25),

$$v(\underline{0}) = L + c^T\underline{S} + u(\underline{S}) \qquad (3.28)$$

$$v(\underline{x}) = v(\underline{0}) - c^T\underline{x} \qquad \underline{x}\varepsilon\sigma \qquad (3.29)$$

Using (3.29) in (3.27), we obtain

$$ng + u(\underline{x}) = L(\underline{x}) + \int_{\underline{x-t}\varepsilon r''(\underline{x})} (n-1)g\,\phi(\underline{t})\,d\underline{t}$$

$$+ \int_{\underline{x-t}\varepsilon\omega(\underline{x})} u(\underline{x-t})\,\phi(\underline{t})\,d\underline{t}$$

$$+ \int_{\underline{x-t}\varepsilon r'(\underline{x})} [v(\underline{0}) - c^T(\underline{x-t})]\,\phi(\underline{t})\,d\underline{t}$$

$$= L(\underline{x}) + (n-1)g\int_{\underline{0}}^{\infty}\phi(\underline{t})\,d\underline{t} + \int_{\underline{x-t}\varepsilon\omega(\underline{x})} u(\underline{x-t})\,\phi(\underline{t})\,d\underline{t}$$

$$+ \int_{\underline{0}}^{\infty} [v(\underline{0}) - c^T(\underline{x-t})]\,\phi(\underline{t})\,d\underline{t}$$

$$- \int_{\underline{x-t}\varepsilon\omega(\underline{x})} [v(\underline{0}) - c^T(\underline{x-t})]\,\phi(\underline{t})\,d\underline{t}$$

Simplifying, we get

$$g + u(\underline{x}) = L(\underline{x}) + v(\underline{0})\,[1 - \int_{\underline{x-t}\varepsilon\omega(\underline{x})}\phi(\underline{t})\,d\underline{t}]$$

$$- \int_{\underline{0}}^{\infty} c^T(\underline{x-t})\,\phi(\underline{t})\,d\underline{t} + \int_{\underline{x-t}\varepsilon\omega(\underline{x})} c^T(\underline{x-t})\,\phi(\underline{t})\,d\underline{t}$$

$$+ \int_{\underline{x-t}\varepsilon\omega(\underline{x})} u(\underline{x-t})\,\phi(\underline{t})\,d\underline{t} \qquad (3.30)$$

Let $\underline{x-t} = \underline{y}$, then (3.30) becomes

$$g + u(\underline{x}) = L(\underline{x}) + v(\underline{0}) \; [1 - \int_{\underline{y}\epsilon\omega(\underline{x})} \phi(\underline{x-y}) \; \underline{dy}]$$

$$- \int_{\underline{0}}^{\infty} c^T \underline{y} \; \phi(\underline{x-y}) \; \underline{dy} + \int_{\underline{y}\epsilon\omega(\underline{x})} c^T \underline{y} \; \phi(\underline{x-y}) \; \underline{dy}$$

$$+ \int_{\underline{y}\epsilon\omega(\underline{x})} u(\underline{y}) \; \phi(\underline{x-y}) \; \underline{dy} \qquad \underline{x}\epsilon\sigma^c \qquad (3.31)$$

Let

$$H(\underline{x}) = L(\underline{x}) - \int_{\underline{0}}^{\infty} c^T \underline{y} \; \phi(\underline{x-y}) \; \underline{dy} + \int_{\underline{y}\epsilon\omega(\underline{x})} c^T \underline{y} \; \phi(\underline{x-y}) \; \underline{dy} \quad \underline{x}\epsilon\sigma^c \quad (3.32)$$

$$\tilde{\phi}(\underline{x}) = \int_{\underline{y}\epsilon\omega(\underline{x})} \phi(\underline{x-y}) \; \underline{dy} \qquad (3.33)$$

On using (3.32) and (3.33) in (3.31), we obtain after transposing g to the R.H.S.

$$u(\underline{x}) = -g + H(\underline{x}) + v(\underline{0}) \; [1 - \tilde{\phi}(\underline{x})]$$

$$+ \int_{\underline{y}\epsilon\omega(\underline{x})} \phi(\underline{x-y}) \; u(\underline{y}) \; \underline{dy} \qquad (3.34)$$

Let

$$A(\underline{x}) = -g + H(\underline{x}) + v(\underline{0}) \; [1 - \tilde{\phi}(\underline{x})] \qquad (3.35)$$

and (3.34) becomes

$$u(\underline{x}) = A(\underline{x}) + \int_{\underline{y}\epsilon\omega(\underline{x})} \phi(\underline{x-y}) \; u(\underline{y}) \; \underline{dy} \qquad \underline{x}\epsilon\sigma^c \qquad (3.36)$$

Each component of $A(\underline{x})$ as given by (3.32) and (3.35) is bounded over the set $\omega(\underline{S})$. Thus, $A(\underline{x})$ is a bounded function over the compact set $\omega(\underline{S})$.

From Theorem 2.1 the solution for (3.36) is given by

$$u(\underline{x}) = A(\underline{x}) + \int_{\underline{y}\in\omega(\underline{x})} \psi(\underline{x-y})\, A(\underline{y})\, d\underline{y} \qquad\qquad \underline{x}\varepsilon\sigma^c \qquad\qquad (3.37)$$

where $\psi(\underline{t}) = \sum_{n=1}^{\infty} \phi_{(n)}(\underline{t})$.

Using (3.35) in (3.37) we get

$$u(\underline{x}) = -g + H(\underline{x}) + v(\underline{0})\, [1 - \tilde{\phi}(\underline{x})]$$

$$+ \int_{\underline{y}\in\omega(\underline{x})} \psi(\underline{x-y})\, [-g + H(\underline{y}) + v(\underline{0})[1 - \tilde{\phi}(\underline{y})]]\, d\underline{y}$$

This last relation is true for all $\underline{x}\varepsilon\sigma^c$. Set $\underline{x} = \underline{S}$ and we get

$$u(\underline{S}) = -g + H(\underline{S}) + v(\underline{0})\, [1 - \tilde{\phi}(\underline{S})]$$

$$+ \int_{\underline{y}\in\omega(\underline{S})} \psi(\underline{S-y})\, [-g + H(\underline{y}) + v(\underline{0})[1 - \tilde{\phi}(\underline{y})]]\, d\underline{y} \qquad (3.38)$$

Using (3.28) in (3.38) we obtain

$$v(\underline{0}) - K - c^T\underline{S} = -g + H(\underline{S}) + v(\underline{0})\, [1 - \tilde{\phi}(\underline{S})]$$

$$+ \int_{\underline{y}\in\omega(\underline{S})} \psi(\underline{S-y})\, [-g + H(\underline{y}) + v(\underline{0})[1 - \tilde{\phi}(\underline{y})]]\, d\underline{y} \qquad (3.39)$$

Let us denote by

$$\Phi_{(n)}(\underline{S}) = \int_{\underline{y}\epsilon\omega(\underline{S})} \phi_{(n)}(\underline{S-y}) \, \underline{dy} \qquad (n \geq 1) \tag{3.40}$$

Note $\phi_{(1)}(\underline{S}) = \tilde{\phi}(\underline{S})$

From (3.39) the coefficient of $v(\underline{0})$ transposed to the R.H.S. is

$$-1 + [1 - \tilde{\phi}(\underline{S})] + \int_{\underline{y}\epsilon\omega(\underline{S})} \psi(\underline{S-y}) \, [1 - \tilde{\phi}(\underline{y})] \, \underline{dy}$$

$$= -\tilde{\phi}(\underline{S}) + \sum_{n=1}^{\infty} \tilde{\Phi}_{(n)}(\underline{S}) - \sum_{n=1}^{\infty} \int_{\underline{y}\epsilon\omega(\underline{S})} \phi_{(n)}(\underline{S-y}) \, \tilde{\phi}(\underline{y}) \, \underline{dy} \tag{3.41}$$

From the definition of $\tilde{\Phi}_{(n)}(\underline{S})$ $(n \geq 1)$, the g-convolution properties, and Remark 2.3, we have for $n \geq 1$

$$\int_{\underline{y}\epsilon\omega(\underline{S})} \phi_{(n)}(\underline{S-y}) \, \tilde{\phi}(\underline{y}) = \int_{\underline{y}\epsilon\omega(\underline{S})} \phi_{(n)}(\underline{S-y}) \, [\int_{\underline{t}\epsilon\omega(\underline{y})} \phi(\underline{y-t}) \, \underline{dt}] \, \underline{dy}$$

$$= \int_{\underline{t}\epsilon\omega(\underline{S})} [\int_{\underline{y}\epsilon\overline{R}_1(\underline{S-t})} \phi_{(n)}(\underline{S-t-y}) \, \phi(\underline{y}) \, \underline{dy}] \, \underline{dt}$$

$$= \int_{\underline{y}\epsilon\omega(\underline{S})} \phi_{(n+1)}(\underline{S-y}) \, \underline{dy} = \Phi_{(n+1)}(\underline{S}) \tag{3.42}$$

Using (3.42) in (3.41) we get

$$-\tilde{\phi}(\underline{S}) + \sum_{n=1}^{\infty} \Phi_{(n)}(\underline{S}) - \sum_{n=1}^{\infty} \Phi_{(n+1)}(\underline{S}) = 0$$

Simplifying (3.39), we get

$$g \, [1 + \int_{\underline{y}\epsilon\omega(\underline{S})} \psi(\underline{S-y}) \, \underline{dy}] = K + c^T\underline{S} + H(\underline{S}) + \int_{\underline{y}\epsilon\omega(\underline{S})} \psi(\underline{S-y}) \, H(\underline{y}) \, \underline{dy}$$

Or,

$$g = \frac{K + \underline{c}^T\underline{S} + H(\underline{S}) + \int_{\underline{y}\epsilon\omega(\underline{S})} \psi(\underline{S-y}) \ H(\underline{y}) \ \underline{dy}}{1 + \int_{\underline{y}\epsilon\omega(\underline{S})} \psi(\underline{S-y}) \ \underline{dy}} \tag{3.43}$$

Let $M = [1 + \int_{\underline{y}\epsilon\omega(\underline{S})} \psi(\underline{S-y}) \ \underline{dy}]^{-1}$. Substituting in (3.43) for $H(\underline{t})$ as

defined by (3.32), gives

$$g = M \{K + \underline{c}^T\underline{S} + L(\underline{S}) + \int_{\underline{y}\epsilon\omega(\underline{S})} \underline{c}^T\underline{y} \ \phi(\underline{S-y}) \ \underline{dy}$$

$$- \int_{\underline{0}}^{\infty} \underline{c}^T\underline{y} \ \phi(\underline{S-y}) \ \underline{dy} + \int_{\underline{y}\epsilon\omega(\underline{S})} \psi(\underline{S-y}) \ [L(\underline{y})$$

$$+ \int_{\underline{t}\epsilon\omega(\underline{y})} \underline{c}^T\underline{t} \ \phi(\underline{y-t}) \ \underline{dt} - \int_{\underline{0}}^{\infty} \underline{c}^T\underline{t} \ \phi(\underline{y-t}) \ \underline{dt}] \ \underline{dy}\}$$

Upon simplifying and arranging terms, we get

$$g = M \{K + c^T\underline{S} + L(\underline{S}) + \int_{\underline{S-y}\epsilon\omega(\underline{S})} \psi(\underline{S-y}) \ L(\underline{y}) \ \underline{dy} - c^T\underline{S}$$

$$+ c^T\underline{u} + \int_{\underline{y}\epsilon\omega(\underline{S})} c^T\underline{y} \ \phi(\underline{S-y}) \ \underline{dy} + \int_{\underline{y}\epsilon\omega(\underline{S})} \psi(\underline{S-y}) \cdot$$

$$[\int_{\underline{t}\epsilon\omega(\underline{y})} c^T\underline{t} \ \phi(\underline{y-t}) \ \underline{dt}] \ \underline{dy} + c^T\mu \int_{\underline{y}\epsilon\omega(\underline{S})} \psi(\underline{S-y}) \ \underline{dy}$$

$$- \int_{\underline{y}\epsilon\omega(\underline{S})} c^T\underline{y} \ \psi(\underline{S-y}) \ \underline{dy} \}$$

where $\underline{\mu}$ is the expectation vector. Simplifying further we get

$$g = M \left\{ K + L(\underline{S}) + \int_{\underline{S-y} \in \omega(\underline{S})} \psi(\underline{S-y}) \, L(\underline{y}) \, \underline{dy} \right\} + c^T \underline{\mu}$$

$$+ M \left\{ \int_{\underline{y} \in \omega(\underline{S})} c^T \underline{y} \, \phi(\underline{S-y}) \, \underline{dy} \right.$$

$$+ \int_{\underline{y} \in \omega(\underline{S})} \psi(\underline{S-y}) \left[\int_{\underline{t} \in \omega(\underline{y})} \phi(\underline{y-t}) \, c^T \underline{t} \, \underline{dt} \right] \underline{dy}$$

$$- \int_{\underline{y} \in \omega(\underline{S})} c^T \underline{y} \, \psi(\underline{S-y}) \, \underline{dy} \right\}$$

$$= M \left\{ K + L(\underline{S}) + \int_{\underline{S-y} \in \omega(\underline{S})} \psi(\underline{S-y}) \, L(\underline{y}) \, \underline{dy} \right\} + c^T \underline{\mu}$$

$$+ M \left\{ \int_{\underline{y} \in \omega(\underline{S})} c^T \underline{y} \, \phi(\underline{S-y}) \, \underline{dy} \right.$$

$$+ \sum_{n=1}^{\infty} \int_{\underline{y} \in \omega(\underline{S})} c^T \underline{y} \phi_{(n+1)} (\underline{S-y}) \, \underline{dy}$$

$$- \sum_{n=1}^{\infty} \int_{\underline{y} \in \omega(\underline{S})} c^T \underline{y} \phi_{(n)} (\underline{S-y}) \, \underline{dy} \right\}$$

Note the last result follows from the definition of $\psi(\underline{t})$, the convolution properties, and Lemma 2.1. Upon simplifying the last relation and substituting for M, (3.26) follows.

Q. E. D.

As given by (3.26) the stationary expected cost per period, g, is a function of $(\sigma, \underline{S})$, and will be written as $g(\sigma, \underline{S})$.

CHAPTER IV

THE OPTIMIZATION PROBLEM

4.1 Introduction

In this chapter we shall provide necessary and sufficient conditions for the existence of an optimal $(\sigma, \underline{S})$ policy that minimizes the expression for $g(\sigma, \underline{S})$ as given by (3.26). The unknowns to be determined are the decision variables S_i ($i = 1, 2, \ldots, m$) and the set Γ.

In Section 4.2 we shall first determine the configuration of the set Γ and then reduce the minimization of $g(\sigma, \underline{S})$ to finding optimal values for $\underline{S}$ and an additional variable C, where C will be identified.

The purpose of Section 4.3 will be to determine the necessary and sufficient conditions for the existence of the pair $(C^*, \underline{S})$ that minimizes $g(\sigma, \underline{S})$. The necessary conditions will be given for the general m-commodity inventory problem, while the sufficient conditions will be given only for the two-commodity problem with seperable $L(x)$. The set of equations satisfied by $(C^*, \underline{S}^*)$ will be restated in terms of a real valued function $M(C^*, \underline{x})$, where $M(C^*, \underline{x})$ will be defined.

Section 4.4 will give a geometrical formulation of the optimization problem in terms of the function $M(C^*, \underline{x})$ and its inherent properties.

Finally in Section 4.5 the linear form for $L(\underline{x})$, the conditional expected holding and shortage cost function, will be considered.

4.2 Characterization of Γ

In this section we shall first characterize the set Γ,

Γ = $\{\underline{x} | \underline{x} \epsilon \Omega; \ Z(\underline{S}-\underline{x}) = 0\}$, by determining explicitly the relation $Z(\underline{S}-\underline{x}) = 0$

up to a constant. This then will be used to reduce the minimization

problem into a problem in differential and integral calculus.

Theorem 4.1:

Under an optimal $(\sigma^*, \underline{S}^*)$ policy, $g(\sigma^*, \underline{S}^*)$, the minimum

stationary total expected cost per period, satisfies the following

relation:

$$g(\sigma^*, \underline{S}^*) = L(\underline{x}^0) + \sigma^T \underline{\mu} \qquad\qquad \underline{x}^0 \epsilon \Gamma^* \qquad\qquad (4.1)$$

where $\underline{\mu}$ is the expectation vector of the random variable $\underline{D}$ with joint

density function $\phi_{\underline{D}}(\underline{t})$.

Proof: The proof is based on the definitions of $h_n(\underline{x})$, $\underline{x} \epsilon \sigma$, and $f_n(\underline{x})$,

$\underline{x} \epsilon \sigma^c$, as given in Chapter III. At optimality we must have for $\underline{x}^0 \epsilon \Gamma^*$

$$h_n(\underline{x}^0) = f_n(\underline{x}^0) \qquad\qquad (n \geq 1)$$

We note that this last relation if explicitly established will determine

the configuration of Γ*.

For large n, $f_n(\underline{x})$ as given by (3.1) can be written

$$f_n(\underline{x}) = L(\underline{x}) + \int_{\underline{x}-\underline{t} \epsilon \omega(\underline{x})} [(n-1)g + u(\underline{x}-\underline{t})] \ \phi(\underline{t}) \ \underline{dt}$$

$$+ \int_{\underline{0}}^{\infty} [(n-1)g + v(\underline{x}-\underline{t})] \ \phi(\underline{t}) \ \underline{dt}$$

$$- \int_{\underline{x-t}\epsilon\omega(\underline{x})} [(n-1)g + v(\underline{x-t})]\phi(\underline{t})\ \underline{dt}\ , \qquad \underline{x}\epsilon\sigma^{c} \qquad\qquad (4.2)$$

where g is the stationary total expected cost per period and u(.) and v(.) are well defined in Chapter III.

For $\underline{x}^0\epsilon\Gamma^*$, and the definition of $\omega(\underline{x})$, the set $\omega(\underline{x}^0)$ is an empty set. Hence, from (4.2) we can write

$$f_n(\underline{x}^0) = L(\underline{x}^0) + \int_{\underline{0}}^{\infty} [(n-1)g + v(\underline{x^0-t})]\ \phi(\underline{t})\ \underline{dt},\ \underline{x}^0\epsilon\Gamma^* \qquad (4.3)$$

using (3.25) and the asymptotic expression for $f_n(\underline{x}^0)$ in (4.3) we obtain

$$ng + K + c^T(\underline{S^*-x}^0) + u(\underline{S^*})$$
$$= L(\underline{x}^0) + \int_{\underline{0}}^{\infty} [(n-1)g + K + c^T(\underline{S^*-x+t}) + u(\underline{S^*})]\ \phi(\underline{t})\ \underline{dt}$$

$$= L(\underline{x}^0) + (n-1)g + K + c^T(\underline{S^*-x}^0) + c^T\underline{\mu} + u(\underline{S^*})$$

simplifying we obtain (4.1).

<div align="right">Q. E. D.</div>

The result of Theorem 4.1, as we shall see later, will be of great importance in finding the necessary and sufficient conditions for the existence of an optimal policy that minimizes the expression for $y(\sigma,\underline{S})$ as given by (3.26). An explicit expression for $L(\underline{x}^0)$, $\underline{x}^0\epsilon\Gamma^*$, can be immediately derived by comparing (3.26) and (4.1) to yield

$$L(\underline{x}^0) = \frac{K + L(\underline{S}^*) + \int_{\underline{S}^*-\underline{t}\varepsilon\omega(\underline{S}^*)} L(\underline{S}^*-\underline{t})\ \psi(\underline{t})\ d\underline{t}}{1 + \int_{\underline{S}^*-\underline{t}\varepsilon\omega(\underline{S}^*)} \psi(\underline{t})\ d\underline{t}} \tag{4.4}$$

which is similar to Iglehart's result [7] for the single commodity case operating under the (s,S) policy. For notational purposes we shall sometimes represent $L(\underline{x}^0)$ by the symbol C^*, C^* refers to the value of the total expected cost per period excluding the variable ordering cost.

Corollary 4.1:

$$\Gamma^* = \{\underline{x}|\underline{x}\varepsilon\Omega;\ L(\underline{x}) - C^* = 0\} \tag{4.5}$$

Proof:

From Section 2.2

$$\Gamma^* = \{\underline{x}|\underline{x}\varepsilon\Omega;\ Z(\underline{S}^*-\underline{x}) = 0\}$$

From Theorem 4.1

$$L(\underline{x}) - C^* = 0 \qquad \underline{x}\varepsilon\Gamma^*$$

Hence,

$$Z(\underline{S}^*-\underline{x}) = L(\underline{x}) - C^* = 0 \qquad \underline{x}\varepsilon\Gamma^*$$

and (4.5) follows.

Q. E. D.

Let us make the transformation $\underline{x} = \underline{S}^*-\underline{t}$; then (Chapter II) the image of the set $\omega(\underline{S}^*)$ is $R(\underline{S}^*)$ and the image of Γ^* is

$$\Gamma_0^* = \{\underline{t}|\underline{t} > \underline{0};\ L(\underline{S}^*-\underline{t}) - C^* = 0\ \} \tag{4.6}$$

The sets $R(\underline{S}^*)$ and $\omega(\underline{S}^*)$ are functions of C^* and often will be referred to as $R(\underline{S}^*,C^*)$ and $\omega(\underline{S}^*,C^*)$.

We may rewrite the expression of $L(\underline{x}^0)$, $\underline{x}^0 \epsilon \Gamma^*$, as

$$L(\underline{x}^0) = \frac{K + L(\underline{S}^*) + \int_{R(\underline{S}^*,C^*)} L(\underline{S}^*-\underline{t}) \, \psi(\underline{t}) \, d\underline{t}}{1 + \int_{R(\underline{S}^*,C^*)} \psi(\underline{t}) \, d\underline{t}} \qquad (4.7)$$

The relation given by (4.1), then, may be written

$$g(\sigma^*,\underline{S}^*) = \frac{K + L(\underline{S}^*) + \int_{R(\underline{S}^*,C^*)} L(\underline{S}^*-\underline{t}) \, \psi(\underline{t}) \, d\underline{t}}{1 + \int_{R(\underline{S}^*,C^*)} \psi(\underline{t}) \, d\underline{t}} + \underline{c}^T \underline{\mu}$$

From this last relation it may be noted that to minimize $g(\sigma,\underline{S})$ relative to the set σ and $\underline{S}$ is equivalent to

$$\text{Min} \left\{ g_1(\underline{S},C) = \frac{K + L(\underline{S}) + \int_{R(\underline{S},C)} L(\underline{S}-\underline{t}) \, \psi(\underline{t}) \, d\underline{t}}{1 + \int_{R(\underline{S},C)} \psi(\underline{t}) \, d\underline{t}} \right\} \qquad (4.8)$$

which is a problem in integral and differential calculus. Let

$$M = \left[1 + \int_{R(S,C)} \psi(\underline{t}) \, d\underline{t} \right]^{-1}, \quad 0 < M < 1,$$

then the minimization problem as given by (4.8) is equivalent to:

$$\min \left\{ g_2(M,C,\underline{S}) = M \left[K + L(\underline{S}) + \int_{R(\underline{S},C)} L(\underline{S}-\underline{t}) \, \psi(\underline{t}) \, d\underline{t} \right] \right\}$$

$$\text{S.T. } \left\{ g_3(M,C,\underline{S}) = M \left[1 + \int_{R(\underline{S},C)} \psi(\underline{t}) \, d\underline{t} \right] - 1 = 0 \right\} \qquad (4.9)$$

4.3 Necessary and Sufficient Conditions

The Lagrange function, $G(M,C,\underline{S},\lambda)$, for (4.9) is given by

$$G(M,C,\underline{S},\lambda) = g_2(M,C,\underline{S}) + \lambda g_3(M,C,\underline{S})$$

$$= M\{K + L(\underline{S}) + \int_{R(\underline{S},C)} L(\underline{S-t})\ \psi(\underline{t})\ \underline{dt}\}$$

$$+ \lambda\{1 - M - M\int_{R(\underline{S},C)} \psi(\underline{t})\ \underline{dt}\ \} \qquad (4.10)$$

By definition a point $(M^*,C^*,\underline{S}^*)$ is a proper relative minimum for $g_2(M,C,\underline{S})$ if

$$g_2(M^*+\Delta M,\ C^*+\Delta C,\ \underline{S^*+\Delta S}) - g_2(M^*,C^*,\underline{S}^*) > 0$$

or equivalently

$$\Delta G = G(M^*+\Delta M,\ C^*+\Delta C,\ \underline{S^*+\Delta S},\lambda^*+\Delta\lambda) - G(M^*,C^*,\underline{S}^*,\lambda^*) > 0 \qquad (4.11)$$

To determine the necessary and sufficient conditions for the existence of a relative minimum for (4.9), we shall first obtain an analytical expression for ΔG as defined in (4.11).

On using (4.10) in (4.11) the expression for ΔG can be written as

$$\Delta G = (M^*+\Delta M)\ \{K + L(\underline{S^*+\Delta S}) + \int_{R(\underline{S^*+\Delta S},C^*+\Delta C)} L(\underline{S^*+\Delta S-t})\ \psi(\underline{t})\ \underline{dt}\ \}$$

$$+ (\lambda^*+\Delta\lambda)\ \{1 - (M^*+\Delta M) - (M^*+\Delta M)\int_{R(\underline{S^*+\Delta S},C^*+\Delta C)} \psi(\underline{t})\ \underline{dt}\ \}$$

$$- M^*\ \{K + L(\underline{S}^*) + \int_{R(\underline{S}^*,C^*)} L(\underline{S^*-t})\ \psi(\underline{t})\ \underline{dt}\ \}$$

$$-\lambda^* \{ 1 - M^* - M^* \int_{R(\underline{S}^*,C^*)} \psi(\underline{t}) \, \underline{dt} \}$$

Upon rearranging terms we obtain

$$\Delta G = M^* \{ L(\underline{S+\Delta S}) - L(\underline{S}^*) + \int_{R(\underline{S^*+\Delta S},C^*+\Delta C)} [L(\underline{S^*+\Delta S-t}) - \lambda^*] \, \psi(\underline{t}) \, \underline{dt}$$

$$- \int_{R(\underline{S}^*,C^*)} [L(\underline{S^*-t}) - \lambda^*] \, \psi(\underline{t}) \, \underline{dt} + \Delta M \{ K + L(\underline{S^*+\Delta S}) - \lambda^*$$

$$+ \int_{R(\underline{S^*+\Delta S},C^*+\Delta C)} [L(\underline{S^*+\Delta S-t}) - \lambda^*] \, \psi(\underline{t}) \, \underline{dt} \}$$

$$+ \Delta\lambda \{ 1 - (M^*+\Delta M) - (M^*+\Delta M) \int_{R(\underline{S^*+\Delta S},C^*+\Delta C)} \psi(\underline{t}) \, \underline{dt} \} \qquad (4.12)$$

If we denote by $\Delta R(\underline{S}^*,C^*)$ the incremental set between $R(\underline{S^*+\Delta S},C^*+\Delta C)$
and $R(\underline{S}^*,C^*)$, then we can write

$$\int_{R(\underline{S^*+\Delta S},C^*+\Delta C)} [L(\underline{S^*+\Delta S-t}) - \lambda^*] \, \psi(\underline{t}) \, \underline{dt}$$

$$- \int_{R(\underline{S}^*,C^*)} [L(\underline{S^*-t}) - \lambda^*] \, \psi(\underline{t}) \, \underline{dt}$$

$$= \int_{R(\underline{S}^*,C^*)} [L(\underline{S^*+\Delta S-t}) - L(\underline{S^*-t})] \, \psi(\underline{t}) \, \underline{dt}$$

$$+ \int_{\Delta R(\underline{S}^*,C^*)} [L(\underline{S^*+\Delta S-t}) - \lambda^*] \, \psi(\underline{t}) \, \underline{dt} \qquad (4.13)$$

and

$$\int_{R(\underline{S}^*+\Delta S,\, C^*+\Delta C)} [L(\underline{S}^*+\Delta S-\underline{t}) - \lambda^*]\ \psi(\underline{t})\ \underline{dt}$$

$$= \int_{R(\underline{S}^*,\, C^*)} [L(\underline{S}^*+\Delta S-\underline{t}) - \lambda^*]\ \psi(\underline{t})\ \underline{dt}$$

$$+ \int_{\Delta R(\underline{S}^*,\, C^*)} [L(\underline{S}^*+\Delta S-\underline{t}) - \lambda^*]\ \psi(\underline{t})\ \underline{dt} \tag{4.14}$$

Using (4.13) and (4.14) in (4.12) we obtain

$$\Delta G = M^* \{ L(\underline{S}^*+\Delta S) - L(\underline{S}^*) + \int_{R(\underline{S}^*,\, C^*)} [L(\underline{S}^*+\Delta S-\underline{t}) - L(\underline{S}^*-\underline{t})]\ \psi(\underline{t})\ \underline{dt}$$

$$+ \int_{\Delta R(\underline{S}^*,\, C^*)} [L(\underline{S}^*+\Delta S-\underline{t}) - \lambda^*]\ \psi(\underline{t})\ \underline{dt} \ \}$$

$$+ \Delta M\ \{ K + L(\underline{S}^*+\Delta S) - \lambda^* [1 + \int_{R(\underline{S}^*,\, C^*)} \psi(\underline{t})\ \underline{dt}]$$

$$+ \int_{R(\underline{S}^*,\, C^*)} L(\underline{S}^*+\Delta S-\underline{t})\ \psi(\underline{t})\ \underline{dt}$$

$$+ \int_{\Delta R(\underline{S}^*,\, C^*)} [L(\underline{S}^*+\Delta S-\underline{t}) - \lambda^*]\ \psi(\underline{t})\ \underline{dt}\}$$

$$+ \Delta\lambda\ \{ 1 - (M^*+\Delta M) - (M^*+\Delta M) \int_{R(\underline{S}^*+\Delta S,\, C^*+\Delta C)} \psi(\underline{t})\ \underline{dt} \ \} \tag{4.15}$$

Let $\underline{t}^0 \epsilon \Delta R(\underline{S}^*,C^*)$ and let $A(\Delta R)$ denotes the area of the incremental set $\Delta R(\underline{S}^*,C^*)$. Then from Theorem 2.2

$$\int_{\Delta R(\underline{S}^*,\, C^*)} [L(\underline{S}^*+\Delta S-\underline{t}) - \lambda^*]\ \psi(\underline{t})\ \underline{dt}$$

$$= [L(\underline{S}^*+\Delta S-\underline{t}^0) - \lambda^*]\ \psi(\underline{t}^0)\ A(\Delta R)$$

using this in (4.15) we obtain

$$\Delta G = M^* \{ L(\underline{S^*+\Delta S}) - L(\underline{S^*})$$

$$+ \int_{R(\underline{S^*},C^*)} [L(\underline{S^*+\Delta S-t}) - L(\underline{S^*-t})] \, \psi(\underline{t}) \, \underline{dt}$$

$$+ [L(\underline{S^*+\Delta S-t}) - \lambda^*] \, \psi(\underline{t}^0) \, A(\Delta R) \} + \Delta M \{ K + L(\underline{S^*+\Delta S})$$

$$- \lambda^*[1 + \int_{R(\underline{S^*},C^*)} \psi(\underline{t}) \, \underline{dt}] + \int_{R(\underline{S^*},C^*)} L(\underline{S^*+\Delta S-t}) \, \psi(\underline{t}) \, \underline{dt}$$

$$+ [L(\underline{S^*+\Delta S-t}^0) - \lambda^*] \, \psi(\underline{t}^0) \, A(\Delta R) \} + \Delta \lambda \{ 1 - (M^*+\Delta M)$$

$$- (M^*+\Delta M) \int_{R(\underline{S^*+\Delta S},C^*+\Delta C)} \psi(\underline{t}) \, \underline{dt} \, \} \qquad (4.16)$$

Let $\underline{t}_0 \epsilon \Gamma_0^*$. Expanding $L(\cdot)$ in a Taylor series, with a remainder, $R^1(\cdot)$, about $(\underline{S^*},C^*)$, the expression for ΔG as given in (4.16) becomes

$$\Delta G = M^* \{ \sum_{i=1}^{m} \frac{\partial L(\underline{S})}{\partial S_i} \Big|_{\underline{S}=\underline{S}^*} \Delta S_i + \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{\partial^2 L(\underline{S})}{\partial S_i \partial S_j} \Big|_{\underline{S}=\underline{S}^*} \Delta S_i \Delta S_j$$

$$+ \int_{R(\underline{S^*},C^*)} [\sum_{i=1}^{m} \frac{\partial L(\underline{S-t})}{\partial S_i} \Big|_{\underline{S}=\underline{S}^*} \Delta S_i$$

$$+ \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{\partial^2 L(\underline{S-t})}{\partial S_i \partial S_j} \Delta S_i \Delta S_j \Big|_{\underline{S}=\underline{S}^*}] \, \psi(\underline{t}) \, \underline{dt}$$

$$+ \{L(\underline{S^*-t_0}) - \lambda^* + \sum_{i=1}^{m} \frac{\partial L(\underline{S-t_0})}{\partial S_i} \bigg|_{\underline{S}=\underline{S}^*} \Delta S_i$$

$$+ \sum_{i=1}^{m} \frac{\partial L(\underline{S^*-t})}{\partial t_i} \bigg|_{\underline{t}=\underline{t}_0} \Delta t_i\} \ \psi(\underline{t}_0) \ A(\Delta R)\} + \Delta M \ \{K + L(\underline{S^*})$$

$$+ \sum_{i=1}^{m} \frac{\partial L(\underline{S})}{\partial S_i} \bigg|_{\underline{S}=\underline{S}^*} \Delta S_i - \lambda^* \ [1 + \int_{R(\underline{S^*},C^*)} \psi(\underline{t}) \ \underline{dt}];$$

$$+ \int_{R(\underline{S^*},C^*)} [L(\underline{S^*-t}) + \sum_{i=1}^{m} \frac{\partial L(\underline{S-t})}{\partial S_i} \bigg|_{\underline{S}=\underline{S}^*} \Delta S_i] \ \psi(\underline{t}) \ \underline{dt}$$

$$+ [L(\underline{S^*-t_0}) - \lambda^* + \sum_{i=1}^{m} \frac{\partial L(\underline{S-t_0})}{\partial S_i} \bigg|_{\underline{S}=\underline{S}^*} \Delta S_i$$

$$+ \sum_{i=1}^{m} \frac{\partial L(\underline{S^*-t})}{\partial t_i} \bigg|_{\underline{t}=\underline{t}_0} \Delta t_i] \ \psi(\underline{t}_0) \ A(\Delta R)\} + \Delta \lambda \ \{1 - (M^*+\Delta M)$$

$$- (M^*+\Delta M) \int_{R(\underline{S^*+\underline{\Delta S}},C^*+\Delta C)} \psi(\underline{t}) \ \underline{dt} \ \}$$

$$+ R^1 (\underline{S^*},C^*,M^*,\lambda^*; \ \underline{S^*+\underline{\Delta S}},C^*+\Delta C,M^*+\Delta M,\lambda^*+\Delta\lambda) \qquad (4.17)$$

The above expression for ΔG will be used to determine the necessary and sufficient conditions for $g_1(\underline{S},C)$ to have a proper relative minimum at $(\underline{S^*},C^*)$.

Necessary Conditions

We shall first find the set of simultaneous conditions that $(\underline{S}^*, C^*)$ will satisfy and then restate these conditions in terms of the function $M(C^*, \underline{x})$ which will be defined later. The following Lemma is needed.

Lemma 4.1:

At optimality

$$\lambda^* = \frac{K + L(\underline{S}^*) + \int_{R(\underline{S}^*, C^*)} L(\underline{S}^*-t)\, \psi(\underline{t})\, \underline{dt}}{1 + \int_{R(\underline{S}^*, C^*)} \psi(\underline{t})\, \underline{dt}} = C^* > 0 \qquad (4.18)$$

Proof: The partial derivative of $G(M, C, \underline{S}, \lambda)$, as given by (4.10), with respect to M, evaluated at $(M^*, C^*, \underline{S}^*, \lambda^*)$, yields

$$\frac{\partial G}{\partial M}\bigg|_{(M^*, C^*, \underline{S}^*, \lambda^*)} = K + L(\underline{S}^*) + \int_{R(\underline{S}^*, C^*)} L(\underline{S}^*-t)\, \psi(\underline{t})\, \underline{dt}$$

$$- \lambda^* [1 + \int_{R(\underline{S}^*, C^*)} \psi(\underline{t})\, \underline{dt}] = 0$$

and

$$\lambda^* = \frac{K + L(\underline{S}^*) + \int_{R(\underline{S}^*, C^*)} L(\underline{S}^*-t)\, \psi(\underline{t})\, \underline{dt}}{1 + \int_{R(\underline{S}^*, C^*)} \psi(\underline{t})\, \underline{dt}}$$

From this last result and (4.7), (4.18) follows.

Q. E. D.

Theorem 4.2:

The optimal values of $C > 0$ and $\underline{S} > \underline{0}$, C^* and $\underline{S}^*$, which

minimize $g_1(\underline{S},C)$, satisfy the following set of equations:

$$\frac{\partial L(\underline{S})}{\partial S_i}\bigg|_{\underline{S}=\underline{S}^*} + \int_{R(\underline{S}^*,C^*)} \frac{\partial L(\underline{S}-t)}{\partial S_i}\bigg|_{\underline{S}=\underline{S}^*} \psi(\underline{t})\, \underline{dt} = 0 \quad (i = 1,2,\ldots,m)$$

$$(4.19)$$

$$C^* - L(\underline{S}^*) + \int_{R(\underline{S}^*,C^*)} [C^* - L(\underline{S}^*-t)]\, \psi(\underline{t})\, \underline{dt} = K \qquad (4.20)$$

Proof:

The partials of $G(M,C,\underline{S},\lambda)$ with respect to S_i $(i = 1, 2, \ldots, m)$

are obtained from (4.17)

$$\frac{\partial G}{\partial S_i}\bigg|_{(M^*,C^*,\underline{S}^*,\lambda^*)} = M^*\left[\frac{\partial L(\underline{S})}{\partial S_i}\bigg|_{\underline{S}=\underline{S}^*}\right.$$

$$+ \int_{R(\underline{S}^*,C^*)} \frac{\partial L(\underline{S}-t)}{\partial S_i}\bigg|_{\underline{S}=\underline{S}^*} \psi(\underline{t})\, \underline{dt}$$

$$\left. + [L(\underline{S}^*-\underline{t}_0) - \lambda^*]\, \psi(\underline{t}_0) \frac{\partial A(\Delta R)}{\partial S_i}\right] = 0 \quad (i = 1, 2,\ldots,m)$$

where the point $\underline{t}_0 \epsilon \Gamma_0^*$. Since by (4.6) and (4.18)

$$\lambda^* = C^* = L(\underline{S}^*-\underline{t}_0), \quad \underline{t}_0 \epsilon \Gamma_0^*,$$

(4.19) follows.

To prove the second part of the theorem we have from (4.18)

$$C^* \left[1 + \int_{R(\underline{S}^*, C^*)} \psi(\underline{t})\, \underline{dt} \right] = K + L(\underline{S}^*) + \int_{R(\underline{S}^*, C^*)} L(\underline{S}^* - \underline{t})\, \psi(\underline{t})\, \underline{dt}$$

And this last relation can be written as

$$C^* - L(\underline{S}^*) + \int_{R(\underline{S}^*, C^*)} [C^* - L(\underline{S}^* - \underline{t})]\, \psi(\underline{t})\, \underline{dt} = K$$

Q. E. D.

Theorem 4.3:

Let the function $M(C^*, \underline{x})$ satisfy the integral equation

$$M(C^*, \underline{x}) = C^* - L(\underline{x}) + \int_{R(\underline{x}, C^*)} M(C^*, \underline{x} - \underline{t})\, \phi(\underline{t})\, \underline{dt} \qquad (4.21)$$

$$= C^* - L(\underline{x}) + \int_{\omega(\underline{x}, C^*)} \phi(\underline{x} - \underline{t})\, M(C^*, \underline{t})\, \underline{dt} \qquad (4.22)$$

Suppose that C^* and $\underline{S}^*$ exist, which minimize $g_1(C, \underline{S})$ subject to $C > 0$ and $\underline{S} > 0$. Then, if $\underline{S}^* > \underline{0}$ and $C^* > 0$, $(C^*, \underline{S}^*)$ is a solution to the set of equations

$$M(C^*, \underline{S}^*) = K \qquad (4.23)$$

$$\frac{\partial M(C^*, \underline{x})}{\partial x_i} \bigg|_{\underline{x} = \underline{S}^*} = 0 \qquad (i = 1, 2, \ldots, m) \qquad (4.24)$$

Proof: Define the functions

$$\hat{M}_i(C^*, \underline{x}) = \frac{\partial L(\underline{x})}{\partial x_i} + \int_{R(\underline{x}, C^*)} \frac{\partial L(\underline{x} - \underline{t})}{\partial x_i}\, \psi(\underline{t})\, \underline{dt} \qquad (i = 1, 2, \ldots, m) \quad (4.25)$$

$$\hat{M}(C^*, \underline{x}) = C^* - L(\underline{x}) + \int_{R(\underline{x}, C^*)} [C^* - L(\underline{x-t})] \ \psi(\underline{t}) \ \underline{dt} \qquad (4.26)$$

$$= C^* - L(\underline{x}) + \int_{\omega(C^*, \underline{x})} [C^* - L(\underline{t})] \ \psi(\underline{x-t}) \ \underline{dt} \qquad (4.27)$$

Let $\underline{x} = \underline{S}^*$ in (4.25) and (4.26) then comparing the results with (4.19) and (4.20) we obtain

$$\hat{M}(C^*, \underline{S}^*) = K \qquad (4.28)$$

$$\hat{M}_i(C^*, \underline{S}^*) = 0 \qquad\qquad (i = 1, 2, \ldots, m) \qquad (4.29)$$

From Theorem 2.1 the solution of (4.22) is given by (4.27). Thus

$$\hat{M}(C^*, \underline{x}) = M(C^*, \underline{x}) \qquad (4.30)$$

Hence from (4.28), (4.23) results which proves the first part of the theorem.

To prove the second part of the theorem we can write from (4.26)

$$\Delta M(C^*, \underline{x}) = M(C^*, \underline{x+\Delta x}) - M(C^*, \underline{x})$$

$$= C^* - L(\underline{x+\Delta x}) + \int_{R(\underline{x+\Delta x}, C^*)} [C^* - L(\underline{x+\Delta x-t})] \ \psi(\underline{t}) \ \underline{dt}$$

$$- [C^* - L(\underline{x}) + \int_{R(\underline{x}, C^*)} [C^* - L(\underline{x-t})] \ \psi(\underline{t}) \ \underline{dt}]$$

If we denote by $\Delta R(\underline{x})$ the incremental set between $R(\underline{x+\Delta x}, C^*)$ and $R(\underline{x}, C^*)$, then the expression for $\Delta M(C^*, \underline{x})$ can be written as

$$\Delta M(C^*,\underline{x}) = - [L(\underline{x+ x}) - L(\underline{x})]$$

$$+ \int_{R(\underline{x},C^*)} [C^* - L(\underline{x+\Delta x-t})] \; \psi(\underline{t}) \; \underline{dt}$$

$$+ \int_{\Delta R(\underline{x})} [C^* - L(\underline{x+\Delta x-t})] \; \psi(\underline{t}) \; \underline{dt}$$

$$- \int_{R(\underline{x},C^*)} [C^* - L(\underline{x-t})] \; \psi(\underline{t}) \; \underline{dt}$$

Let $\underline{t}^0 \epsilon \Delta R(\underline{x})$ and let $A(\Delta R(\underline{x}))$ denotes the area of $\Delta R(\underline{x})$. Using the Mean Value Theorem for multiple integrals, Theorem 2.2, we can write

$$\Delta M(C^*,\underline{x}) = - [L(\underline{x+\Delta x}) - L(\underline{x})] - \int_{R(\underline{x},C^*)} [L(\underline{x+\Delta x-t}) - L(\underline{x-t})] \psi(\underline{t}) \; \underline{dt}$$

$$+ [C^* - L(\underline{x+\Delta x-t}^0)] \; \psi(\underline{t}^0) \; A(\Delta R(\underline{x})) \qquad (4.31)$$

Expanding $L(\underline{x+\Delta x})$, $L(\underline{x+\Delta x-t}^0)$ in a Taylor series, with a remainder, $R^1(\underline{x+\Delta x,x};C^*)$, about $\underline{x}$, the expression for $\Delta M(C^*,\underline{x})$ becomes

$$\Delta M(C^*,\underline{x}) = - \sum_{i=1}^{m} \frac{\partial L(\underline{x})}{\partial x_i} \Delta x_i - \int_{R(\underline{x},C^*)} \sum_{i=1}^{m} \frac{\partial L(\underline{x-t})}{\partial x_i} \Delta x_i \; \psi(\underline{t}) \; \underline{dt}$$

$$+ [C^* - L(\underline{x-t}^0) - \sum_{i=1}^{m} \frac{\partial L(\underline{x-t}^0)}{\partial x_i} \Delta x_i] \; \psi(\underline{t}^0) \; A(\Delta R(\underline{x}))$$

$$+ R^1(\underline{x+\Delta x,x};C^*)$$

As $\Delta\underline{x} \to \underline{0}$, $\underline{t}^0 \to \underline{t}_0 \epsilon \Gamma_0(\underline{x})$. Hence $\lim_{\Delta\underline{x}\to 0} \{L(\underline{x}-\underline{t}^0) - C^*\} = 0$. Dividing

$\Delta M(C^*,\underline{x})$ by Δx_i and letting $\Delta x_i \to 0$ we get

$$\frac{\partial M(C^*,\underline{x})}{\partial x_i} = - \left\{ \frac{\partial L(\underline{x})}{\partial x_i} + \int_{R(\underline{x},C^*)} \frac{\partial L(\underline{x}-\underline{t})}{\partial x_i} \psi(\underline{t}) \, \underline{dt} \right\}$$

$$(i = 1, 2, \ldots, m) \qquad (4.32)$$

Comparing (4.25) and (4.32) we obtain

$$\frac{\partial M(C^*,\underline{x})}{\partial x_i} = - M_i(C^*,\underline{x}) \qquad (i = 1, 2, \ldots, m)$$

Hence from (4.29), (4.24) follows.

Q. E. D.

<u>Sufficient Conditions</u>

So far the necessary conditions for the determination of the optimal values of C and $\underline{S}$ were discussed. To study the sufficient conditions that have to be satisfied, we shall assume that the function L($\underline{x}$) is separable, that is,

$$L(\underline{x}) = \sum_{i=1}^{m} L_i(x_i)$$

where $L_i(x_i)$ for all i is twice differentiable. Now, since $g_3(M^*+\Delta M, C^*+\Delta C, \underline{S}^*+\Delta\underline{S}) = 0$, and since the necessary conditions, derived above, require that each derivative of $G(M,C,\underline{S},\lambda)$ vanish (4.17) reduces to

$$\Delta G = G(M^*+\Delta M, C^*+\Delta C, \underline{S^*+\Delta S}, \lambda^*+\Delta\lambda) - G(M^*, C^*, \underline{S}^*, \lambda^*)$$

$$= M^* \left[\frac{1}{2} \sum_{j=1}^{m} \sum_{i=1}^{m} \frac{\partial L(\underline{S}-\underline{t})}{\partial S_i \partial S_j} \Bigg|_{\underline{S}=\underline{S}^*} \Delta S_i \Delta S_j \right.$$

$$+ \int_{R(\underline{S}^*, C^*)} \frac{1}{2} \sum_{j=1}^{m} \sum_{i=1}^{m} \frac{\partial L(\underline{S}-\underline{t})}{\partial S_i \partial S_j} \Bigg|_{\underline{S}=\underline{S}^*} \Delta S_i \Delta S_j \ \psi(\underline{t}) \ d\underline{t}$$

$$+ \left\{ L(\underline{S^*-t}_0) - \lambda^* + \sum_{i=1}^{m} \frac{\partial L(\underline{S-t}_0)}{\partial S_i} \Bigg|_{\underline{S}=\underline{S}^*} \Delta S_i \right.$$

$$+ \sum_{i=1}^{m} \frac{\partial L(\underline{S^*-t})}{\partial t_i} \Bigg|_{\underline{t}=\underline{t}_0} \Delta t_i \left. \right\} \psi(\underline{t}_0) \ A(\Delta R) \right]$$

$$+ R^1(\underline{S}^*, C^*, M^*, \lambda^* ; \underline{S^*+\Delta S}, C^*+\Delta C, M^*+\Delta M, \lambda^*+\Delta\lambda) \qquad (4.33)$$

where $\underline{t}_0 = (t_{10}, t_{20}, \ldots, t_{mo}) \ \epsilon\Gamma_0^*$, $R^1(\cdot)$ is the remainder terms

of the Taylor series expansion, $A(\Delta R)$ is the area of $\Delta R(\underline{S}^*, C^*)$, and

$L'_i(t_i)$, $L''_i(t_i)$, $(i = 1, 2, \ldots, m)$, are the first and second derivatives

of $L_i(t_i)$.

Since $L(\underline{S^*-t}_0) = \lambda^* = C^*$, we have from (4.33) on taking the

second partial derivatives, for $i, j = 1, 2, \ldots, m$

$$\frac{\partial^2 G}{\partial S_i^2} \Bigg|_{(M^*, C^*, \underline{S}^*, \lambda^*)} = M^* [L_i''(S_i^*) + \int_{R(\underline{S}^*, C^*)} L_i''(S_i^* - t_i) \ \psi(\underline{t}) \ d\underline{t}$$

$$+ L_i'(S_i^* - t_{i0}) \ \psi(\underline{t}_0) \frac{\partial A(\Delta R)}{\partial S_i} \Bigg|_{(C^*, \underline{S}^*)}] \qquad (4.34)$$

$$\frac{\partial^2 G}{\partial S_i \partial S_j}\bigg|_{(M^*, C^*, \underline{S}^*, \lambda^*)} = M^* \left[L_i'(S_i^* - t_{i0}) \ \psi(\underline{t}_0) \ \frac{\partial A(\Delta R)}{\partial S_i}\bigg|_{(C^*, \underline{S}^*)} \right] (i \neq j) \quad (4.35)$$

$$\frac{\partial^2 G}{\partial S_i \partial C}\bigg|_{(M^*, C^*, \underline{S}^*, \lambda^*)} = M^* \left[L_i'(S_i^* - t_{i0}) \ \psi(\underline{t}_0) \ \frac{\partial A(\Delta R)}{\partial C}\bigg|_{(C^*, \underline{S}^*)} \right] \quad (4.36)$$

$$\frac{\partial^2 G}{\partial C^2}\bigg|_{(M^*, C^*, \underline{S}^*, \lambda^*)} = M^* \left[\psi(t_0) \ \frac{\partial A(\Delta R)}{\partial C}\bigg|_{(C^*, \underline{S}^*)} \right] \quad (4.37)$$

Note the value of $\underline{t}_0$ is not the same in (4.34), (4.35), (4.36), and (4.37). A sufficient condition for (4.9) to have a proper minimum a $(M^*, C^*, \underline{S}^*)$ is that the matrix $\hat{A}$ given below, is positive definite.

$$\hat{A} = \begin{vmatrix} \dfrac{\partial^2 G}{\partial C^2} & \dfrac{\partial^2 G}{\partial C \partial S_1} & \dfrac{\partial^2 G}{\partial C \partial S_2} & \cdots & \dfrac{\partial^2 G}{\partial C \partial S_m} \\[2ex] \dfrac{\partial^2 G}{\partial C \partial S_1} & \dfrac{\partial^2 G}{\partial S_1^2} & \dfrac{\partial^2 G}{\partial S_1 \partial S_2} & \cdots & \dfrac{\partial^2 G}{\partial S_1 \partial S_m} \\[2ex] \dfrac{\partial^2 G}{\partial C \partial S_2} & \dfrac{\partial^2 G}{\partial S_1 \partial S_2} & \dfrac{\partial^2 G}{\partial S_2^2} & \cdots & \dfrac{\partial^2 G}{\partial S_2 \partial S_m} \\[2ex] \vdots & & & & \\[1ex] \dfrac{\partial^2 G}{\partial C \partial S_m} & \dfrac{\partial^2 G}{\partial S_1 \partial S_m} & \dfrac{\partial^2 G}{\partial S_2 \partial S_m} & \cdots & \dfrac{\partial^2 G}{\partial S_m^2} \end{vmatrix}$$

where each element of $\hat{A}$ is evaluated at the point $(M^*, C^*, \underline{S}^*, \lambda^*)$.

The next two theorems will provide the sufficient conditions for the two-commodity problem to have a proper minimum at (C^*, S_1^*, S_2^*).

Theorem 4.4:

If $(C^*, \underline{S}^*)$ is a solution to the set of equations (4.19) and (4.20), then $(C^*, \underline{S}^*)$ is a relative minima of $g_1(C, \underline{S})$ if

$$L_i'(S_i^* - t_{i0}) < 0, \quad (i=1,2), \quad \forall \underline{t}_0 = (t_{10}, t_{20}) \varepsilon \Gamma_0^*$$

and

$$L_i''(S_i^*) + \int_{R(\underline{S}^*, C^*)} L_i''(S_i^* - t_i) \, \psi(\underline{t}) \, \underline{dt}$$

$$+ L_i'(S_i^* - t_i^*) \, \psi(\underline{t}^*) \, \frac{\partial A(\Delta R)}{\partial S_i} \bigg|_{(C^*, \underline{S}^*)} > 0 \qquad (i=1,2) \qquad (4.38)$$

where the point $\underline{t}^*$, $\underline{t}^* = (t_1^*, t_2^*) \epsilon \Gamma_0^*$, has different values for (i=1,2).

<u>Proof</u>: For the two commodity problem we must have

(a) $\dfrac{\partial^2 G}{\partial C^2} > 0$, and this implies from (4.37) that

$$\dfrac{\partial A(\Delta R)}{\partial C} \bigg|_{(C^*, \underline{S}^*)} > 0 \tag{4.39}$$

(b) $\dfrac{\partial^2 G}{\partial C^2}\ \dfrac{\partial^2 G}{\partial S_1^2} - \left[\dfrac{\partial^2 G}{\partial C \partial S_1}\right]^2 > 0$ and from (a) this implies

$$\dfrac{\partial^2 G}{\partial S_1^2} \bigg|_{(M^*, C^*, \underline{S}^*, \lambda^*)} > 0 \tag{4.40}$$

Hence from (4.34), since $M^* > 0$, (4.38) for $i = 1$ follows.

(c) $\dfrac{\partial^2 G}{\partial S_2^2} \left\{ \dfrac{\partial^2 G}{\partial C^2} \cdot \dfrac{\partial^2 G}{\partial S_1^2} - \left(\dfrac{\partial^2 G}{\partial C \partial S_1}\right)^2 \right\}$

$\qquad - \dfrac{\partial^2 G}{\partial S_1 \partial S_2} \left\{ \dfrac{\partial^2 G}{\partial C^2} \cdot \dfrac{\partial^2 G}{\partial S_1 \partial S_2} - \dfrac{\partial^2 G}{\partial C \partial S_1} \cdot \dfrac{\partial^2 G}{\partial C \partial S_2} \right\}$

$\qquad + \dfrac{\partial^2 G}{\partial C \partial S_2} \left\{ \dfrac{\partial^2 G}{\partial C \partial S_1} \cdot \dfrac{\partial^2 G}{\partial S_1 \partial S_2} - \dfrac{\partial^2 G}{\partial C \partial S_2} \cdot \dfrac{\partial^2 G}{\partial S_1^2} \right\} > 0$

Upon regrouping terms we obtain

$$\frac{\partial^2 G}{\partial S_2{}^2} \left\{ \frac{\partial^2 G}{\partial C^2} \cdot \frac{\partial^2 G}{\partial S_1{}^2} - \left(\frac{\partial^2 G}{\partial C \partial S_1} \right)^2 \right\}$$

$$+ 2 \frac{\partial^2 G}{\partial S_1 \partial S_2} \cdot \frac{\partial^2 G}{\partial C \partial S_1} \cdot \frac{\partial^2 G}{\partial C \partial S_2} - \frac{\partial^2 G}{\partial C^2} \cdot \left(\frac{\partial^2 G}{\partial S_1 \partial S_2} \right)^2$$

$$- \frac{\partial^2 G}{\partial S_1{}^2} \cdot \left(\frac{\partial^2 G}{\partial C \partial S_2} \right)^2 > 0$$

Upon using (a) and (b) we get

$$\frac{\partial^2 G}{\partial S_2{}^2} \left\{ \frac{\partial^2 G}{\partial C^2} \cdot \frac{\partial^2 G}{\partial S_1{}^2} - \left(\frac{\partial^2 G}{\partial C \partial S_1} \right)^2 \right\} + 2 \frac{\partial^2 G}{\partial S_1 \partial S_2} \cdot \frac{\partial^2 G}{\partial C \partial S_1} \cdot \frac{\partial^2 G}{\partial C \partial S_2} > 0 \quad (4.41)$$

Now a sufficient condition for

$$\left. \frac{\partial A(\Delta R)}{\partial C} \right|_{(C^*, \underline{S}^*)} > 0$$

is that, for any $\underline{t}_0 \varepsilon \Gamma_0{}^*$, $L_i'(S_i^* - t_{i0}) < 0$ $(i=1,2)$. The proof proceeds

as follows:

$$\Gamma_0{}^* = \{ \underline{t}_0 \,|\, L_1(S_1^* - t_{10}) + L_2(S_2^* - t_{20}) = C^* \}$$

In particular if $A = (t_{10}, t_{20}) \varepsilon \Gamma_0{}^*$, Fig. (4.1), then

$$C^* = L_1(S_1^* - t_{10}) + L_2(S_2^* - t_{20})$$

If we increment C by ΔC, then for $B(t_{10}, t_{20} + \Delta t_{20})$ and $D(t_{10} + \Delta t_{10}, t_{20})$

$$C^* + \Delta C = L_1(S_1^* - t_{10}) + L_2(S_2^* - (t_{20} + \Delta t_{20}))$$

$$= L_1(S_1^* - (t_{10} + \Delta t_{10})) + L_2(S_2^* - t_{20})$$

and

$$\Delta C = -L_2'(S_2^* - t_{20})\Delta t_{20} = -L_1'(S_1^* - t_{10})\Delta t_{10}$$

$$\frac{\Delta C}{\Delta t_{20}} = -L_2'(S_2^* - t_{20}) \; ; \quad \frac{\Delta C}{\Delta t_{10}} = -L_1'(S_1^* - t_{10})$$

Thus

$$\frac{\Delta C}{\Delta t_{20}} > 0 \text{ if } L_2'(S_2^* - t_{20}) < 0; \quad \frac{\Delta C}{\Delta t_{10}} > 0 \text{ if } L_1'(S_1^* - t_{10}) < 0$$

And hence

$$\left. \frac{\partial A(\Delta R)}{\partial C} \right|_{(C^*, \underline{S}^*)} > 0$$

if $L_i'(S_i - t_{i0}) < 0$ (i=1,2) for all $\underline{t}_0 \epsilon \Gamma_0^*$, which proves the first

part of the theorem.

Using this condition and the fact that

$$\left. \frac{\partial A(\Delta R)}{\partial S_i} \right|_{(C^*, \underline{S}^*)} > 0 \qquad (i=1,2)$$

we get from (4.35) and (4.36)

$$\frac{\partial^2 G}{\partial S_1 \partial S_2}\bigg|_{(M^*,C^*,\underline{S}^*,\lambda^*)} < 0; \quad \frac{\partial^2 G}{\partial C \partial S_1}\bigg|_{(M^*,C^*,\underline{S}^*,\lambda^*)} < 0;$$

$$\frac{\partial^2 G}{\partial C \partial S_2}\bigg|_{(M^*,C^*,\underline{S}^*,\lambda^*)} < 0 \tag{4.42}$$
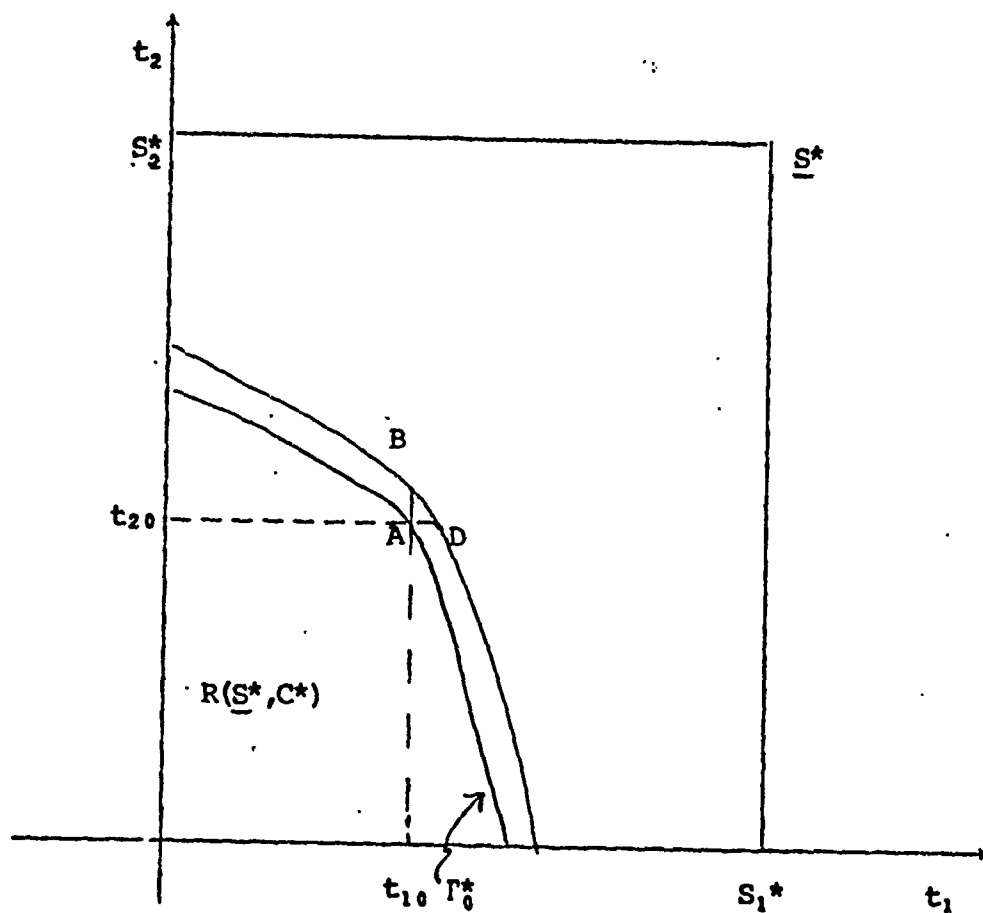
Fig. (4.1)

Hence, from (4.42)

$$2\frac{\partial^2 G}{\partial S_1 \partial S_2} \cdot \frac{\partial^2 G}{\partial C \partial S_1} \cdot \frac{\partial^2 G}{\partial C \partial S_2} < 0$$

and (4.41) implies that

$$\frac{\partial^2 G}{\partial S_2^2} \left\{ \frac{\partial^2 G}{\partial C^2} \cdot \frac{\partial^2 G}{\partial S_1^2} - \left(\frac{\partial^2 G}{\partial C \partial S_1} \right)^2 \right\} > 0 \qquad (4.43)$$

From (b) we have

$$\frac{\partial^2 G}{\partial C^2} \cdot \frac{\partial^2 G}{\partial S_1^2} - \left(\frac{\partial^2 G}{\partial C \partial S_1} \right)^2 > 0$$

Therefore from (4.43) it follows that

$$\frac{\partial^2 G}{\partial S_2^2} \Bigg|_{(M^*,C^*,\underline{S}^*,\lambda^*)} > 0$$

and (4.38) for i=2 is proved.

$$Q. E. D.$$

Theorem 4.5:

If $(\underline{S}, C^*)$ is a solution to the set of equations (4.19) and (4.20), then $(\underline{S}^*, C^*)$ is a proper relative minima of $g_1(\underline{S}, C)$ if

(i) $L_i'(S_i^* - t_{i0}) < 0 \quad \forall \underline{t}_0 = (t_{10}, t_{20}) \varepsilon \Gamma_0^* \quad (i=1,2)$

(ii)   M(C*,$\underline{x}$) is a strictly concave function of $\underline{x}$, where

M(C*,$\underline{x}$) is given by (4.21)

Proof:

Expanding L($\underline{x+\Delta x}$), L($\underline{x+\Delta x-t^0}$) in a Taylor series with a remainder R^2($\underline{x+\Delta x, x}$;C*), about $\underline{x}$, the expression for ΔM, in E^2, as given by (4.31) becomes

$$\Delta M(C^*,\underline{x}) = - \{ \sum_{i=1}^{2} L_i'(x_i)\Delta x_i + \sum_{i=1}^{2} L_i''(x_i)\Delta x_i^2 \}$$

$$- \{ \int_{R(\underline{x},C^*)} [\sum_{i=1}^{2} L_i'(x_i-t_i)\Delta x_i$$

$$+ \sum_{i=1}^{2} L_i''(x_i-t_i)\Delta x_i^2] \psi(\underline{t}) \, d\underline{t}\} + \{C^* - L(\underline{x-t^0})$$

$$- \sum_{i=1}^{2} L_i'(x_i-t_i^0)\Delta x_i\} \psi(\underline{t^0}) A(\Delta R(\underline{x}))$$

$$+ R^2(\underline{x, x+\Delta x};C^*) \tag{4.44}$$

where $\Delta R(\underline{x})$ is the incremental set between R($\underline{x+\Delta x}$,C*) and R($\underline{x}$;C*), A($\Delta R(\underline{x})$) is the area of $\Delta R(\underline{x})$, and $\underline{t^0}\epsilon\Delta R(\underline{x})$.

Let $\underline{t_0} = (t_{10}, t_{20})\epsilon\Gamma_0(\underline{x})$. Then from (4.44) we can write

$$\frac{\partial^2 M(C^*,\underline{x})}{\partial x_i^2} = - \{L_i''(x_i) + \int_{R(\underline{x},C^*)} L_i''(x_i-t_i) \psi(\underline{t}) \, d\underline{t}$$

$$+ L_i'(x_i - t_{i_0}) \; \psi(\underline{t}_0) \; \frac{\partial A(\Delta R(\underline{x}))}{\partial x_i} \; \bigg|_{(\underline{x}, C^*)} \} \quad (i=1,2) \qquad (4.45)$$

$$\frac{\partial^2 M(C^*, \underline{x})}{\partial x_i \partial x_j} = -L_i'(x_i - t_{i_0}) \; \psi(\underline{t}_0) \; \frac{\partial A(\Delta R(\underline{x}))}{\partial x_j} \; \bigg|_{(\underline{x}, C^*)} \quad i \neq j \;\; (i,j=1,2) \qquad (4.46)$$

Now, $M(C^*, \underline{x})$ is a strictly concave function of $\underline{x}$ if and only if the Hessian is negative definite. Hence we must have

$$\text{(a)} \; \frac{\partial^2 M(C^*, \underline{x})}{\partial x_1^2} < 0; \quad \text{(b)} \; \begin{vmatrix} \dfrac{\partial^2 M(C^*, \underline{x})}{\partial x_1^2} & \dfrac{\partial^2 M(C^*, \underline{x})}{\partial x_1 \partial x_2} \\[3mm] \dfrac{\partial^2 M(C^*, \underline{x})}{\partial x_1 \partial x_2} & \dfrac{\partial^2 M(C^*, \underline{x})}{\partial x_2^2} \end{vmatrix} > 0$$

or, from (4.45) for $i=1$,

$$\text{(a)} \; L_1''(x_1) + \int_{R(\underline{x}, C^*)} L_1''(x_1 - t_1) \; \psi(\underline{t}) \; \underline{dt}$$

$$+ L_1'(x_1 - t_{10}) \; \psi(\underline{t}_0) \; \frac{\partial A(\Delta R(\underline{x}))}{\partial x_1} \; \bigg|_{(\underline{x}, C^*)} > 0 \qquad (4.47)$$

and

$$\text{(b)} \; \frac{\partial^2 M(C^*, \underline{x})}{\partial x_1^2} \cdot \frac{\partial^2 M(C^*, \underline{x})}{\partial x_2^2} - \left\{ \frac{\partial^2 M(C^*, \underline{x})}{\partial x_1 \partial x_2} \right\}^2 > 0$$

from which

$$\frac{\partial^2 M(C^*,\underline{x})}{\partial x_1{}^2} \cdot \frac{\partial^2 M(C^*,\underline{x})}{\partial x_2{}^2} > 0$$

Hence, from (a) and (4.45) for i = 2, we have

$$L_2{}''(x_1) + \int_{R(\underline{x},C^*)} L_2{}''(x_2-t_2) \ \psi(\underline{t}) \ \underline{dt}$$

$$+ L_2{}'(x_2-t_{20}) \ \psi(\underline{t}_0) \ \frac{\partial A(\Delta R(\underline{x}))}{\partial x_2} \bigg|_{(\underline{x},C^*)} > 0 \qquad (4.48)$$

Set $\underline{x} = \underline{S}^*$ in (4.47) or (4.48), then for (i = 1,2)

$$L_i{}''(S_i{}^*) + \int_{R(\underline{S}^*,C^*)} L_i{}''(S_i{}^*-t_i) \ \psi(\underline{t}) \ \underline{dt}$$

$$+ L_i{}'(S_i{}^*-t_{i0}) \ \psi(\underline{t}_0) \ \frac{\partial A(\Delta R)}{\partial S_i} \bigg|_{(\underline{S}^*,C^*)}$$

$$> 0, \ \underline{t}_0 = (t_{10}, \ t_{20}) \epsilon \Gamma_0{}^*$$

Therefore, condition (4.38) will hold if $M(C^*,\underline{x})$ is a strictly concave function of $\underline{x}$.

Q. E. D.

Unfortunately, as stated, the sufficient conditions are not useable in practice.

4.4   Geometric Reformulation of the Problem

At this point it is of interest to use the inherent characteristics of the function $M(C^*, \underline{x})$ in giving geometric reformulation of the problem for determining the local minima at the point $(\underline{S}^*, C^*)$ in E^2.

If $\underline{S}^*$ and C^* satisfy the necessary and sufficient conditions for a proper relative minima, then as we have seen in Section 4.3 determining $\underline{S}^*$ and C^* is equivalent to constructing the function $M(C^*, \underline{x})$ whose equation is given by (4.22) with the following properties:

(a)   for $\underline{x}^0 \varepsilon \Gamma^*$

$$M(C^*, \underline{x}^0) = 0$$

$$\frac{\partial M(C^*, \underline{x})}{\partial x_i} \bigg|_{\underline{x} = \underline{x}^0} = -L_i{}'(x_i{}^0) \qquad (i = 1, 2)$$

(b) $M(C^*, \underline{x})$ achieves a maximum value of K at $\underline{x} = \underline{S}^*$,

i.e., $M(C^*, \underline{S}^*) = K$

Graphically, this reformulation is illustrated in Fig. (4.2). $M(C^*, \underline{x})$ is plotted for various values of the parameter C^*. The function
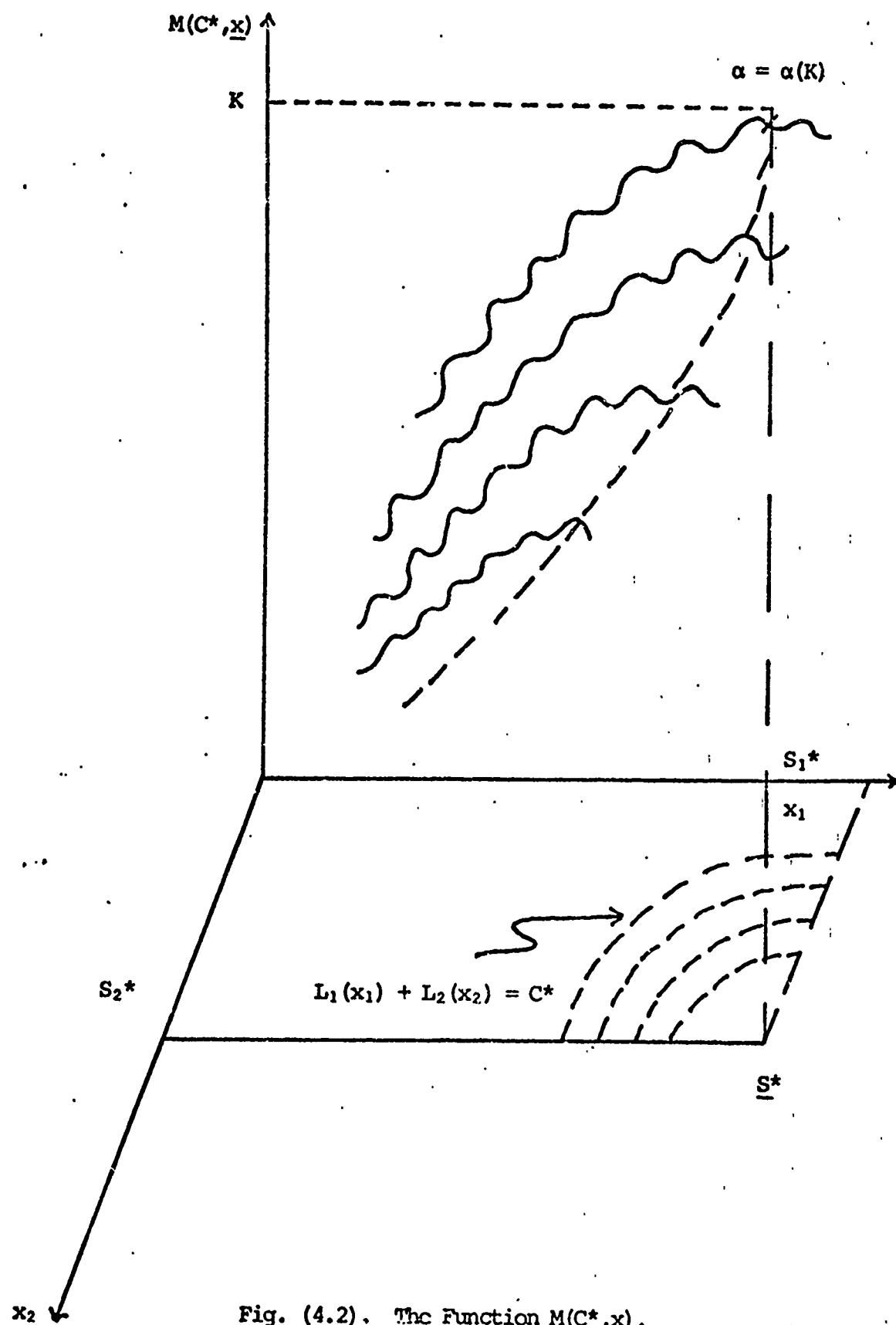
Fig. (4.2). The Function $M(C^*, \underline{x})$.

$\alpha = \alpha(K)$ is the locus of all points $\underline{S}^*$ at which the function $M(C^*,\underline{x})$ attains a maximum value of K for various values of C^*.

4.5 The Function $L(\underline{x})$

From now on we shall consider the case when $L(\underline{x})$ has the form

$$L(\underline{x}) = \sum_{i=1}^{m} L_i(x_i)$$

where each $L_i(x_i)$ is continuous and twice differentiable. The function $L(\underline{x})$ is the cost charged over a given period of time excluding the ordering cost and in general it is the holding and shortage costs.

Consider the case when the holding and shortage costs for each item are linear. Let for item $(i = 1, 2, \ldots, m)$

h_i = Unit holding cost at the end of the period

p_i = Unit shortage cost at the end of the period

Then $L(\underline{x})$ is the total expected holding and shortage costs measured at the end of the period and is given by

$$L(\underline{x}) = \sum_{i=1}^{m} h_i \int_0^\infty \cdots \int_0^{x_i} \cdots \int_0^\infty (x_i - t_i)\, \phi(t_1, \ldots, t_i, \ldots, t_m) dt_1 \ldots dt_i \ldots dt_m$$

$$+ \sum_{i=1}^{m} p_i \int_0^\infty \cdots \int_{x_i}^\infty \cdots \int_0^\infty (t_i - x_i)\, \phi(t_1, \ldots, t_i, \ldots, t_m)\, dt_1 \ldots dt_i \ldots dt_m$$

$$(4.49)$$

If $\phi_i(\cdot)$ and $\Phi_i(\cdot)$ denote respectively the marginal density function and marginal distribution function of the demand for item i ($i = 1, 2, \ldots, m$) in a given period, then from (4.49) we obtain

$$L(\underline{x}) = \sum_{i=1}^{m} \left[h_i \int_0^{x_i} (x_i - t_i) \, \phi_i(t_i) \, dt_i + p_i \int_{x_i}^{\infty} (t_i - x_i) \, \phi_i(t_i) \, dt_i \right]$$

(4.50)

Setting

$$L_i(x_i) = h_i \int_0^{x_i} (x_i - t_i) \, \phi_i(t_i) \, dt_i + p_i \int_{x_i}^{\infty} (t_i - x_i) \, \phi_i(t_i) \, dt_i$$

$$= (h_i + p_i) \int_0^{x_i} \Phi_i(t_i) \, dt_i + p_i(\mu_i - x_i) \qquad (i = 1, 2, \ldots, m)$$

(4.51)

we get from (4.50)

$$L(\underline{x}) = \sum_{i=1}^{m} L_i(x_i)$$

$L_i(x_i)$ denotes the marginal expected holding and shortage costs for item i measured at the end of a period. From (4.51) it is clear that the function $L_i(x_i)$ is twice differentiable at all points for which $\phi_i(x_i)$ is continuous. In fact for ($i = 1, 2, \ldots, m$)

$$L_i'(x_i) = (h_i + p_i)\phi_i(x_i) - p_i$$

$$L_i''(x_i) = (h_i + p_i)\phi_i(x_i)$$

and $L_i(x_i)$ is strictly convex for all points x_i for which $\phi_i(x_i) > 0$.

Hence, $L(\underline{x})$, being the sum of strictly convex functions, is itself a strictly convex function.

CHAPTER V

COMPUTATIONAL ASPECTS OF THE PROBLEM - AN EXAMPLE

5.1 Introduction

In this chapter the analysis shall be restricted, to the computational aspects of the optimization problem, for the special case of a two-commodity problem where the demand for the items obeys the exponential distribution, and the holding and shortage costs are linear.

In Section 5.2 we will present the two-commodity problem under consideration and obtain explicit expression for the function $L_i(x_i)$, $(i = 1,2)$, as given by (4.51), and its first and second derivatives.

The purpose of Section 5.3 will be to determine analytical expressions for the set of equations used to determine the optimal policy parameters. The integral equation (4.22) will be converted into a partial differential equation of the second order. Riemann method will, be used to solve this boundary value problem at the point $\underline{S}$.

Finally in Section 5.4 a numerical example will be considered. Numerical methods will be used to solve for $S_1{}^*$, $S_2{}^*$, and C^*.

5.2 Example

Consider a two-commodity inventory control problem where the demand, $\{D_j\}$, for the items over a sequence of periods ($j = 1,2,\ldots$), is assumed to be independently and identically distributed continuous random variables with joint density function

$$\phi_{\underline{D}}(\underline{t}) = \phi(\underline{t}) = \lambda_1\lambda_2 e^{-\lambda_1 t_1 - \lambda_2 t_2}$$

The marginal probability distribution function is given by

$$\Phi_i(t_i) = \int_0^{t_i} \phi_i(t)\ dt = 1 - e^{-\lambda_i t_i}$$

From (4.51) we can write

$$L_i(x_i) = (h_i + p_i) \int_0^{x_i} \Phi_i(t_i)\ dt_i + p_i(\mu_i - x_i)$$

$$= (h_i + p_i) \int_0^{x_i} (1 - e^{-\lambda_i t_i})\ dt_i + p_i\left(\frac{1}{\lambda_i} - x_i\right)$$

$$= (h_i + p_i)\left(x_i + \frac{1}{\lambda_i}\ e^{-\lambda_i x_i} - \frac{1}{\lambda_i}\right) + p_i\left(\frac{1}{\lambda_i} - x_i\right)$$

$$= \frac{(h_i + p_i)}{\lambda_i}\ e^{-\lambda_i x_i} + h_i x_i - \frac{h_i}{\lambda_i} \qquad (i = 1,2) \qquad (5.1)$$

On taking the first and second derivatives of (5.1), we get

$$L_i'(x_i) = -(h_i + p_i)\ e^{-\lambda_i x_i} + h_i \qquad (i = 1,2) \qquad (5.2)$$

$$L_i''(x) = \lambda_i(h_i+p_i)\, e^{-\lambda_i x_i} > 0 \qquad (i=1,2) \tag{5.3}$$

Using (5.1) in (4.5) we get

$$\Gamma^* = \{\underline{x}\mid\underline{x}\in\Omega;\ h_1 x_1 + h_2 x_2 + \frac{h_1+p_1}{\lambda_1}e^{-\lambda_1 x_1} + \frac{h_2'+p_2}{\lambda_2}e^{-\lambda_2 x_2}$$

$$= C^* + \frac{h_1}{\lambda_1} + \frac{h_2}{\lambda_2}\} \tag{5.4}$$

5.3 Computational Aspects

From Theorem 4.2, S_1^*, S_2^*, and C^* are the solutions for the set of equations given by (4.19) and (4.20). Using (5.1) and (5.2) in (4.19) and (4.20) we get

$$h_i - (h_i+p_i)\, e^{-\lambda_i S_i^*} + \int_{R(\underline{S}^*,C^*)} [h_i - (h_i+p_i)\cdot$$

$$e^{-\lambda_i(S_i^*-t_i)}]\, \psi(\underline{t})\, \underline{dt} = 0 \qquad (i=1,2) \tag{5.5}$$

$$C^* - \sum_{i=1}^{2}\,[\,\frac{(h_i+p_i)}{\lambda_i}e^{-\lambda_i S_i^*} + h_i S_i^* - \frac{h_i}{\lambda_i}\,] + \int_{R(\underline{S}^*,C^*)}[C^*$$

$$-\sum_{i=1}^{2}\,[\,\frac{(h_i+p_i)}{\lambda_i}e^{-\lambda_i(S_i^*-t_i)} + h_i(S_i^*-t_i)$$

$$-\frac{h_i}{\lambda_i}\,]\, \psi(\underline{t})\, \underline{dt} = K \tag{5.6}$$

where

$$\psi(\underline{t}) = \sum_{n=1}^{\infty} \phi_{(n)}(\underline{t})$$

From (5.5) we can write

$$[h_i - (h_i + p_i) e^{-\lambda_i S_i^*}] = \frac{\int_{R(\underline{S}^*, C^*)} (h_i + p_i) e^{-\lambda_i t_i} \psi(\underline{t}) \, \underline{dt}}{1 + \int_{R(\underline{S}^*, C^*)} \psi(\underline{t}) \, \underline{dt}} \qquad (i=1,2)$$

Or

$$S_i^* = \frac{1}{\lambda_i} \ln \left[\frac{\int_{R(\underline{S}^*, C^*)} e^{-\lambda_i t_i} \psi(\underline{t}) \, \underline{dt}}{\int_{R(\underline{S}^*, C^*)} \psi(\underline{t}) \, \underline{dt}} - \frac{h_i}{h_i + p_i} \right] \qquad (i=1,2)$$

The complexity of finding an analytical expression for $\psi(\underline{t})$ dictates abandoning the direct solution of (5.5) and (5.6) for S_1^*, S_2^*, and C^*.

Now from Theorem 4.3 the values of $\underline{S}^*$ and C^* are the real positive solutions to the set of simultaneous equations given by (4.23) and (4.24), i.e.

$$M(C^*, \underline{S}^*) = K; \quad \frac{\partial M(C^*, \underline{x})}{\partial x_i} \bigg|_{\underline{x} = \underline{S}^*} = 0 \qquad (i=1,2) \qquad (5.7)$$

where $M(C^*, \underline{x})$ is given by (4.21).

To solve the set of equations given by (5.7), we shall first find an expression for $M(C^*,\underline{x})$ and then use this expression to determine S_1^*, S_2^*, and C^*. One way to solve for $M(C^*,\underline{x})$ is to convert the integral equation (4.22) into a partial differential equation of the second order.

<u>Theorem 5.1:</u>

In E^2 if

$$\phi(\underline{t}) = \lambda_1 \lambda_2 \, e^{-\lambda_1 t_1 - \lambda_2 t_2} \qquad (\lambda_i > 0, \; t_i \geq 0; \; i=1,2)$$

and $L(\underline{t}) = L_1(t_1) + L_2(t_2)$, where $L_i(t_i)$, i=1,2, is given by (5.1), then $M(C^*,\underline{x})$ satisfying the integral equation (4.22), i.e.

$$M(C^*,\underline{x}) = C^* - L(\underline{x}) + \int_{\omega(\underline{x},C^*)} \phi(\underline{x-t}) \, M(C^*,\underline{t}) \, \underline{dt} \qquad (5.8)$$

is the solution for the boundary value problem

$$\frac{\partial^2 M(C^*,\underline{x})}{\partial x_1 \partial x_2} + \lambda_2 \frac{\partial M(C^*,\underline{x})}{\partial x_1} + \lambda_1 \frac{\partial M(C^*,\underline{x})}{\partial x_2} = \lambda_1 \lambda_2 \, [C^* - h_1 x_1 - h_2 x_2]$$

and for $\underline{x}^0 \epsilon \Gamma^*$

$$M(C^*,\underline{x}^0) = 0$$

$$\left. \frac{\partial M(C^*,\underline{x})}{\partial x_i} \right|_{\underline{x}=\underline{x}^0} = -L_i{'}(x_i{}^0) = (h_i + p_i) e^{-\lambda_j x_i{}^0} - h_i \qquad (i=1,2)$$

(5.9)

Proof:

Before deriving the partial differential equation, we will note that for $i,j=1,2$

$$\frac{\partial \phi(\underline{t})}{\partial t_i} = -\lambda_i \phi(\underline{t}) \tag{5.10}$$

$$\frac{\partial^2 \phi(\underline{t})}{\partial t_i \partial t_j} = \lambda_i \lambda_j \phi(\underline{t}) \tag{5.11}$$

The partial differential equation will be derived directly from the integral equation using again the method of variable increment on ΔM as in Chapter IV.

$$\Delta M = M(C^*, \underline{x+\Delta x}) - M(C^*, \underline{x})$$

$$= C^* - L(\underline{x+\Delta x}) + \int_{\omega(\underline{x+\Delta x}, C^*)} \phi(\underline{x+\Delta x-t}) \, M(\underline{t}) \, \underline{dt}$$

$$- [C^* - L(\underline{x}) + \int_{\omega(\underline{x}, C^*)} \phi(\underline{x-t}) \, M(\underline{t}) \, \underline{dt}]$$

adding and subtracting $\int_{\omega(\underline{x}, C^*)} \phi(\underline{x+\Delta x-t}) \, M(\underline{t}) \, \underline{dt}$ to the right hand

side we get after simplification

$$\Delta M = -\{L(\underline{x+\Delta x}) - L(\underline{x})\} + \int_{\omega(\underline{x}, C^*)} [\phi(\underline{x+\Delta x-t}) - \phi(\underline{x-t})] \, M(\underline{t}) \, \underline{dt}$$

$$+ \int_{\omega(\underline{x+\Delta x},C^*)} \phi\underline{(x+\Delta x-t)} \ M(\underline{t}) \ \underline{dt}$$

$$- \int_{\omega(\underline{x},C^*)} \phi\underline{(x+\Delta x-t)} \ M(\underline{t}) \ \underline{dt} \qquad (5.12)$$

Let $\Delta\omega(\underline{x},C^*)$ denote the incremental set between $\omega(\underline{x+\Delta x},C^*)$ and $\omega(\underline{x},C^*)$, $A(\Delta\omega(\underline{x}))$ denote the area of $\Delta\omega(\underline{x})$, and $\underline{t}^0\varepsilon\Delta\omega(\underline{x})$. Using the Mean Value Theorem for multiple integrals, (Theorem 2.2), we can write

$$\int_{\omega(\underline{x+\Delta x},C^*)} \phi\underline{(x+\Delta x-t)} \ M(\underline{t}) \ \underline{dt} - \int_{\omega(\underline{x},C^*)} \phi\underline{(x+\Delta x-t)} \ M(\underline{t}) \ \underline{dt}$$

$$= \phi\underline{(x+\Delta x-t}^0) \ M(\underline{t}^0) \ A(\Delta\omega(\underline{x})) \qquad (5.13)$$

Using (5.13) in (5.12) we obtain

$$\Delta M = - \ \{L\underline{(x+\Delta x)} - L(\underline{x})\} + \int_{\omega(\underline{x},C^*)} [\phi\underline{(x+\Delta x-t)} - \phi\underline{(x-t)}] \ M(\underline{t}) \ \underline{dt}$$

$$+ \ \phi\underline{(x+\Delta x-t}^0) \ M(\underline{t}^0) \ A(\Delta\omega(\underline{x}))$$

Expanding $L\underline{(x+\Delta x)}$ and $\phi\underline{(x+\Delta x-t}^0)$ in a Taylor Series about the point $\underline{x}$, the expression for ΔM can be written as

$$\Delta M = - \ \{ \sum_{i=1}^{2} L_i{}'(x_i)\Delta x_i + \sum_{i=1}^{2} L_i{}''(x_i)\Delta x_i{}^2 \}$$

$$+ \int_{\omega(\underline{x},C^*)} [\ \sum_{i=1}^{2} \frac{\partial\phi\underline{(x-t)}}{\partial x_i} \Delta x_i$$

$$+ \frac{1}{2} \sum_{j=1}^{2} \sum_{i=1}^{2} \frac{\partial^2 \phi \, (\underline{x-t})}{\partial x_i \partial x_j} \, \Delta x_i \wedge x_j \;] \; M(\underline{t}) \; \underline{dt} + \{\phi \, (\underline{x-t}^0)$$

$$+ \sum_{i=1}^{2} \frac{\partial \phi \, (\underline{x-t}^0)}{\partial x_i} \, \Delta x_i \} \; M(\underline{t}^0) \; A(\Delta \omega \, (\underline{x})) \} + R^1 \, (\underline{x+\Delta x}, \underline{x}; C^*) \tag{5.14}$$

where $R^1 \, (\underline{x+\Delta x}, \underline{x}; C^*)$ denotes the remainder term of the expansion. On using (5.2), (5.3), (5.10), and (5.11) in (5.14) we get

$$\Delta M = \{ (h_1 + p_1) e^{-\lambda_1 x_1} - h_1 \} \Delta x_1 - \{ (h_2 + p_2) e^{-\lambda_2 x_2} - h_2 \} \; \Delta x_2$$

$$- \lambda_1 (h_1 + p_1) e^{-\lambda_1 x_1} \Delta x_1^{\,2} - \lambda_2 (h_2 + p_2) e^{-\lambda_2 x_2} \Delta x_2^{\,2}$$

$$+ \int_{\omega(\underline{x}, C^*)} [\; -\lambda_1 \phi \, (\underline{x-t}) \Delta x_1 \; - \; \lambda_2 \phi \, (\underline{x-t}) \Delta x_2$$

$$+ \lambda_1^{\,2} \phi \, (\underline{x-t}) \Delta x_1^2 + \lambda_1 \lambda_2 \phi \, (\underline{x-t}) \Delta x_1 \Delta x_2 + \lambda_2^{\,2} \phi \, (\underline{x-t}) \Delta x_2^{\,2}] \; \cdot$$

$$M(\underline{t}) \; \underline{dt} + \{\phi \, (\underline{x-t}^0) \; - \; \lambda_1 \phi \, (\underline{x-t}^0) \Delta x_1 \; - \; \lambda_2 \phi \, (\underline{x-t}^0) \Delta x_2 \} \; \cdot$$

$$M(\underline{t}^0) \; A(\Delta \omega \, (\underline{x})) + R^1 \, (\underline{x+\Delta x}, \underline{x}; C^*) \tag{5.15}$$

Figure (5.1) shows geometrically the incremental areas due to positive increments in x_1 and x_2.

Now from (5.15) we have for $i=1,2$

$$\frac{\partial M(C^*, \underline{x})}{\partial x_i} = (h_i + p_i) e^{-\lambda_i x_i} - h_i - \lambda_i \int_{\omega(\underline{x}, C^*)} \phi \, (\underline{x-t}) \; M(\underline{t}) \; \underline{dt}$$

$$+ \phi(\underline{x-t}_0) \ M(\underline{t}_0) \ \frac{\partial A(\Delta\omega(\underline{x}))}{\partial x_i} \qquad\qquad \underline{t}_0 \epsilon \Delta\omega(\Delta x_i)$$

or explicitly

$$\frac{\partial M(C^*,\underline{x})}{\partial x_1} = (h_1+p_1)e^{-\lambda_1 x_1} - h_1 - \lambda_1 \int_{\omega(\underline{x},C^*)} \phi(\underline{x-t}) \ M(\underline{t}) \ \underline{dt}$$

$$+ \phi(0,x_2-t_{20}) \ M(x_1,t_{20}) \ \frac{\partial A(\Delta\omega(\underline{x}))}{\partial x_1} \qquad (5.16)$$

where, from Fig. (5.1), $t_{20} \epsilon \ [\hat{x}_2,x_2]$.

$$\frac{\partial M(C^*,\underline{x})}{\partial x_2} = (h_2+p_2)e^{-\lambda_2 x_2} - h_2 - \lambda_2 \int_{\omega(\underline{x},C^*)} \phi(\underline{x-t}) \ M(\underline{t}) \ \underline{dt}$$

$$+ \phi(x_1-t_{10},0) \ M(t_{10},x_2) \ \frac{\partial A(\Delta\omega(\underline{x}))}{\Delta x_2} \qquad (5.17)$$

where, from Fig. (5.1), $t_{10} \epsilon [\hat{x}_1,x_1]$.

$$\frac{\partial^2 M(C^*,\underline{x})}{\partial x_1 \partial x_2} = \lambda_1 \lambda_2 \int_{\omega(\underline{x},C^*)} \phi(\underline{x-t}) \ M(\underline{t}) \ \underline{dt}$$

$$- \lambda_2 \phi(0,x_2-t_{20}) \ M(x_1,t_{20}) \ \frac{\partial A(\Delta\omega(\underline{x}))}{\partial x_1}$$

$$- \lambda_1 \psi(x_1-t_{10},0) \ M(t_{10},x_2) \ \frac{\partial A(\Delta\omega(\underline{x}))}{\partial x_2}$$
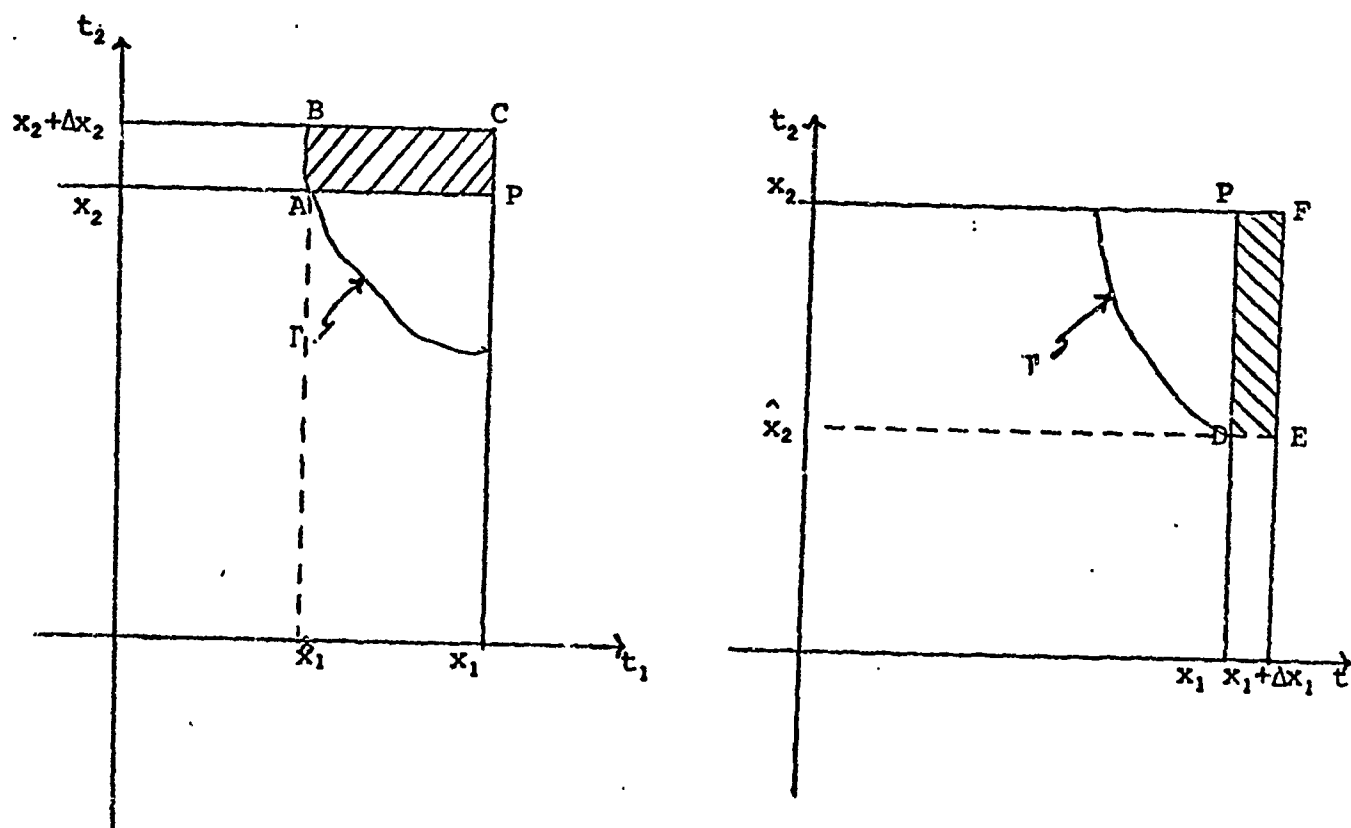
Fig. (5.1)

$$+ \phi(0,0) \ M(x_1,x_2) \tag{5.18}$$

where from Fig. (5.1), $t_{10} \epsilon [\hat{x}_1, x_1]$ and $t_2 \epsilon [\hat{x}_2, x_\bullet]$. Using (5.16) and

(5.17) and the explicit expression for $M(\underline{x}) = M(C^*, \underline{x})$ in (5.18), we get:

$$\frac{\partial^2 M(C^*, \underline{x})}{\partial x_1 \partial x_2} = \lambda_1 \lambda_2 \int_{\omega(\underline{x}, C^*)} \phi(\underline{x-t}) \ M(\underline{t}) \ \underline{dt}$$

$$- \lambda_2 \phi(0, x_2 - t_{20}) \ M(x_1, t_{20}) \ \frac{\partial A(\Delta \dot{\omega}(\underline{x}))}{\partial x_2}$$

$$- \lambda_1 \phi(x_1 - t_{10}, 0) \ M(t_{10}, x_2) \ \frac{\partial A(\Delta \omega(\underline{x}))}{\partial x_2}$$

$$+ \lambda_1 \lambda_2 \ \{ C^* - [\frac{(h_1 + p_1)}{\lambda_1} e^{-\lambda_1 x_1} + h_1 x_1 - \frac{h_1}{\lambda_1}$$

$$+ \frac{(h_2 + p_2)}{\lambda_2} e^{-\lambda_2 x_2} + h_2 x_2 - \frac{h_2}{\lambda_2}]$$

$$+ \int_{\omega(\underline{x}, C^*)} \phi(\underline{x-t}) \ M(\underline{t}) \ \underline{dt} \ \}$$

$$= -\lambda_2 \frac{\partial M(C^*, \underline{x})}{\partial x_1} + \lambda_2 \ \{ (h_1 + p_1) e^{-\lambda_1 x_1} - h_1 \}$$

$$- \lambda_1 \frac{\partial M(C^*, \underline{x})}{\partial x_2} + \lambda_1 \ \{ (h_2 + p_2) e^{-\lambda_2 x_2} - h_2 \}$$

$$+ \lambda_1 \lambda_2 \ \{ C^* - [\frac{(h_1 + p_1)}{\lambda_1} e^{-\lambda_1 x_1} + h_1 x_2 - \frac{h_1}{\lambda_1}$$

$$+ \frac{(h_2 + p_2)}{\lambda_2} \; e^{-\lambda_2 x_2} + h_2 x_2 - \frac{h_2'}{\lambda_2}]\}$$

$$= - \lambda_2 \frac{\partial M(C^*, \underline{x})}{\partial x_1} - \lambda_1 \frac{\partial M(C^*, \underline{x})}{\partial x_2} + \lambda_2 \lambda_2 [C^* - h_1 x_1 - h_2 x_2]$$

Hence the partial differential equation given in (5.9) follows. Since for $\underline{x}^0 \varepsilon \Gamma^*$ the set $\omega(\underline{x}^0)$ is an empty set, we obtain from (5.8)

$$M(C^*, \underline{x}^0) = C^* - L(\underline{x}^0) = 0$$

and

$$\left. \frac{\partial M(C^*, \underline{x})}{\partial x_i} \right|_{\underline{x} = \underline{x}} = L_i' \; (x_i^0) \qquad (i = 1, 2)$$

On substituting for $L_i' (x_i^0)$ as given by (5.2) the boundary conditions follow.

Q. E. E.

Theorem 5.2:

The solution of the boundary value problem (5.9) at the point $\underline{S}$ is given by

$$M(C^*, \underline{S}) = \int_{\Gamma^*} e^{-\lambda_1 (S_1 - x_1) - \lambda_2 (S_2 - x_2)} \; I_0 [2\sqrt{\lambda_1 \lambda_2 (S_1 - x_1)(S_2 - x_2)} \,]$$

$$[(h_1 + p_1) \, e^{-\lambda_1 x_1} - h_1] \; dx_1$$

$$+ \lambda_1 \lambda_2 \int_{\omega(C^*, \underline{S})} [C^* - h_1 x_1 - h_2 x_2] \cdot$$

$$e^{-\lambda_1(S_1-x_1) - \lambda_2(S_2-x_2)} I_0 [2\sqrt{\lambda_1 \lambda_2 (S_1-x_1)(S_2-x_2)}] \, dx_1 dx_2 \quad (5.19)$$

where $I_0[\ ]$ is the modified Bessel function of the first kind of zero order.

<u>Proof</u>:

In solving the boundary problem, which is a linear hyperbolic partial differential equation of the second order, we will use a method due to Riemann (see [29]). The value of the function at the point $\underline{S}$ is given by

$$M(C^*, \underline{S}) = \frac{1}{\lambda_1 \lambda_2} \left\{ \int_{\Gamma^*} U(\underline{S}, \underline{x}) \frac{\partial M(C^*, \underline{x})}{\partial x_1} dx_1 \right.$$

$$\left. + \int_{\omega(C^*, \underline{S})} \lambda_1 \lambda_2 [C^* - h_1 x_1 - h_2 x_2] \, U(\underline{S}, \underline{x}) \, \underline{dx} \right\} \quad (5.20)$$

where $U(\underline{S}, \underline{x})$ is the Green's function for the problem satisfying the following boundary problem

$$\frac{\partial^2 U}{\partial x_1 \partial x_2} - \lambda_2 \frac{\partial U}{\partial x_1} - \lambda_1 \frac{\partial U}{\partial x_2} = 0$$

$$\frac{\partial U}{\partial x_1} = \lambda_1 U \quad \text{when } x_2 = S_2 \quad\quad\quad (5.21)$$

$$\frac{\partial U}{\partial x_2} = \lambda_2 U \quad \text{when} \quad x_1 = S_1$$

$$U = \lambda_1 \lambda_2 \quad \text{when} \quad x_1 = S_1, \quad x_2 = S_2$$

(5.21)

A solution for (5.21) is given by (see [25])

$$U(\underline{x}, \underline{S}) = \lambda_1 \lambda_2 \; e^{-\lambda_1 (S_1 - x_1) - \lambda_2 (S_2 - x_2)} \; I_0 [2\sqrt{\lambda_1 \lambda_2 (S_1 - x_1)(S_2 - x_2)}]$$

(5.22)

using in (5.20) expression (5.22) and the value of

$$\left. \frac{\partial M(C^*, \underline{x})}{\partial x_i} \right|_{\underline{x} \in \Gamma^*} , \quad i = 1, 2$$

as given in (5.9), we obtain (5.19)

Q. E. D.

5.4 Numerical Example

In this numerical example, we shall make use of the parameter values considered by Sivazlian [13]. Let us consider the case when

$$\lambda_1 = \lambda_2 = 1$$

$$p_1 = p_2 = 20$$

$$h_1 = h_2 = 1$$

$$K = 5.0$$

Using these values in (5.1), (5.2), (5.4), and (5.19) we get

$$L_i(x_i) = +21e^{-x_i} + x_i - 1 \qquad (i=1,2)$$

$$L_i'(x_i) = -21e^{-x_i} + 1 \qquad (i=1,2)$$

$$\Gamma^* = \{\underline{x} \mid x_1 + 21e^{-x_1} + x_2 + 21e^{-x_2} = C^* + 2\}$$

and at the point $\underline{S} = (S_1, S_2)$

$$
M(C^*, \underline{S}) = \int_{\Gamma^*} e^{-(S_1 - x_1) - (S_2 - x_2)} I_0[2\sqrt{(S_1 - x_1)(S_2 - x_2)}] \cdot
$$

$$
\{21e^{-x_1} - 1\}\, dx_1 + \int_{\omega(C^*, \underline{S})} (C^* - x_1 - x_\_) \cdot
$$

$$
e^{-(S_1 - x_1) - (S_2 - x_2)} I_0[2\sqrt{(S_1 - x_1)(S_2 - x_2)}]\, \underline{dx} \qquad (5.23)
$$

Fig. (5.2) illustrates geometrically the configuration of the set $\omega(C^* \underline{S})$ for the case when $C^* = 8$. To determine the values of S_1^*, S_2^*, and C^*, we have to solve the set of simultaneous equations given by

(5.7), where $M(C^*, \underline{x}) \Big|_{\underline{x} = \underline{S}}$ is given by (5.23). The complexity of

obtaining an analytical expression for $M(C^*, \underline{S})$ is evident and therefore numerical results are sought.

$L_1(x_1) + L_2(x_2) = C^*$ for the case when $C^* = 8$

$\omega(C^*, \underline{S})$

$\Gamma^* = \{\underline{x} \mid x_1 + x_2 + 21e^{-x_1} + 21e^{-x_2} = 10\}$
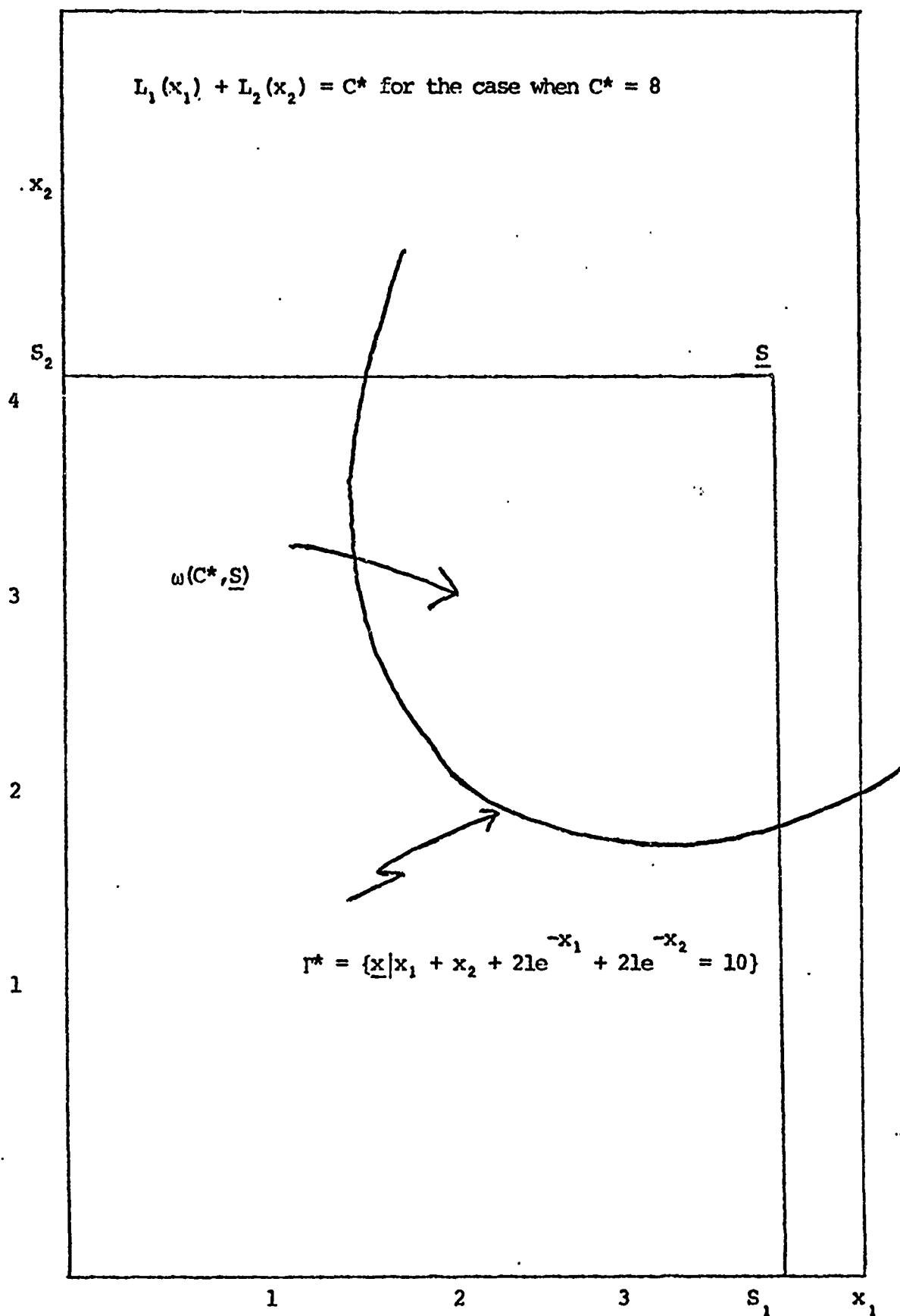
Fig. (5.2)

Numerical Analysis:

In carrying out the numerical analysis for this particular example, the following points were taken into consideration:

(i) at optimality $S_1^* = S_2^*$

(ii) for a given value of C^*, Γ^* is symmetric about the line $x_1 = x_2$

(iii) the results obtained by Sivazlian [13], table 1, for this particular example under different (σ, S) policies

(iv) Sivazlian [13] shows that

$$\int_0^a \int_0^a e^{-(u_1 + u_2)} I_0[2\sqrt{u_1 u_2}] \, du_1 \, du_2$$

$$= a\{1 - [e^{-2a} I_0(2a) + e^{-2a} I_1(2a)]\} \quad (a > 0)$$

(v) Sivazlian [13] further shows that

$$\int_0^a \int_0^a u_i e^{-(u_1 + u_2)} I_0[2\sqrt{u_1 u_2}] \, du_1 \, du_2$$

$$= \frac{a[1 + 2a]}{2} \{1 - [e^{-2a} I_0(2a) + e^{-2a} I_1(2a)]\}$$

$$(i = 1, 2)$$

(vi) for this particular example, $M(C^*, \underline{x})$, as given in (5.8) is a non-decreasing function of C^*.

Taking the above points into consideration, we initiated the search procedure by choosing $C^* = 8.2$ and $S_1 = S_2 = 3.8$. In computing the double integral in (5.23) for a particular value of C^*, S_1 and S_2, the region $\omega(C^*,\underline{S})$ is partitioned into four smaller regions as shown in Fig. (5.3). The coordinates of the point P_1 are $(S_1-1.8, S_2-1.8)$.

(1) Using (iv) and (v) we have

$$\int_{A_1} (C^*-x_1-x_2)e^{-(S_1-x_1) - (S_2-x_2)} I_0[2\sqrt{(S_1-x_1)(S_2-x_2)}]\ \underline{dx}$$

$$= (C^*-S_1-S_2)\ [1.8\{1 - [e^{-3.6}\ I_0(3.6) + e^{-3.6}\ I_1(3.6)]\}]$$

$$+ 2 \cdot \frac{1.8[1 + 3.6]}{2} \{1 - [e^{-3.6}\ I_0(3.6)$$

$$+ e^{-3.6}\ I_1(3.6)]\} - \frac{(1.8)^2}{2}$$

$$= 1.07109288\ (C^*-S_1-S_2) + 1.68702725$$

where the values of $e^{-3.6}\ I_0(3.6)$ and $e^{-3.6}\ I_1(3.6)$ were taken from Bessel function tables [19].

(2) The double integral of the function

$$(C^*-x_1-x_2)e^{-(S_1-x_1) - (S_2-x_2)} I_0[2\sqrt{(S_1-x_1)(S_2-x_2)}],$$
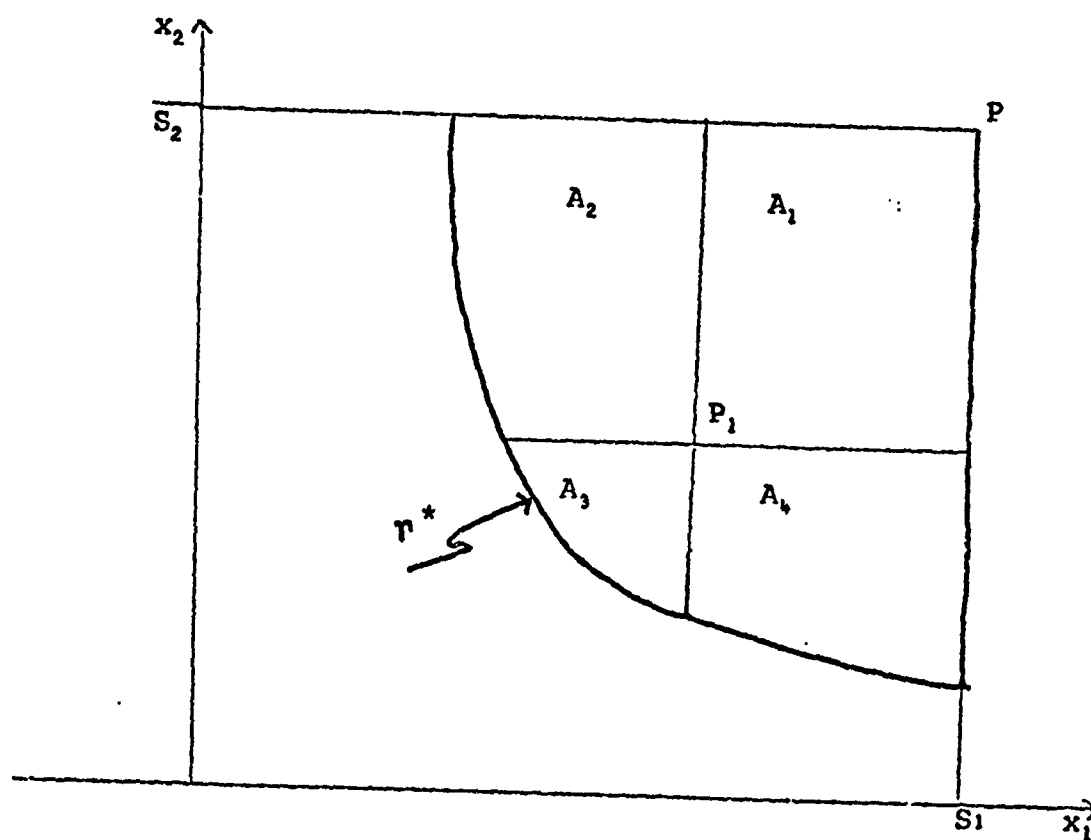
Fig. (5.3)

which is defined over the closed subregions A_2, A_3

and A_4 is formulated by cutting each subregion into

very small squares each of area ΔA_i, $(i = 2,3,4)$,

where any two of these squares have at most one side

in common. Hence for $(i = 2,3,4)$

$$\int_{A_i} (C^* - x_1 - x_2)\, e^{-(S_1-x_1)-(S_2-x_2)} I_0[2\sqrt{(S_1-x_1)(S_2-x_2)}]\, \underline{dx}$$

$$= \lim_{\Delta A_i \to 0} \sum_{j=1}^{n_i} (C^* - x_{1_j} - x_{2_j})\, e^{-(S_1-x_{1_j})-(S_2-x_{2_j})}$$

$$I_0[2\sqrt{(S_1 - x_{1_j})(S_2 - x_{2_j})}]\, \Delta A_i$$

where (x_{1_j}, x_{2_j}) is any point in A_i and n_i is the maximum

number of squares of area ΔA_i in A_i. By symmetry, the

integral over A_4 is equal to the integral over A_2.

(3)   The line integral of the function

$$e^{-(S_1-x_1)-(S_2-x_2)} I_0[2\sqrt{(S_1 - x_1)(S_2 - x_2)}] \cdot \{21e^{-X} - 1\}$$

which is defined over the smooth arc Γ^* is formulated

by dividing the arc Γ^* into N arcs by inserting $(N - 1)$

points $Q_i(x_{1i}, x_{2i})$. Thus,

$$\int_{\Gamma^*} e^{-(S_1-x_1)-(S_2-x_2)} I_0[2\ (S_1 - x_1)(S_2 - x_2)]\{21e^{-X_1}-1]\}\, dx_1$$

$$= \lim_{\substack{N \to \infty \\ \Delta x_{1i} \to 0 \\ \Delta x_{2i} \to 0}} \sum_{i=1}^{N} e^{-(S_1 - x_{1i}) - (S_2 - x_{2i})} I_0 [2\sqrt{(S_1 - x_{1i})(S_2 - x_{2i})}] \{21e^{-x_{1i}} - 1\} \Delta x_{1i}$$

Numerical Results:

To obtain numerical results, a computational algorithm in Fortran IV (appendix 5.A) was written to solve for the optimal policy variables. The algorithm proceeds in the following steps:

Step 0 – Choose initial values for C^* and the increment in $C^*, \Delta C^*$;

     – Choose initial values for $\underline{S}$, $\underline{S} = \underline{S}_0$, and an increment in $\underline{S}$, $\Delta \underline{S}$. The range of values of $\underline{S}$, $\underline{S}_0 \leq \underline{S} \leq \underline{S}^0$, for a given value of C^*, are chosen to assure that $M(C^*, \underline{S})$ achieves a maximum in that range;

     – Choose ΔK as an upper limit on $K - M(C^*, \underline{S})$.

Step 1 – Compute $M(C^*, \underline{S})$.

Step 2 – Check the difference between $M(C^*, \underline{S})$ and K, then

     a. If the difference is greater than ΔK, set

$$\Delta C^* = \frac{\Delta C^*}{2}, \quad C^* = C^* - \Delta C^*, \quad \underline{S} = \underline{S}_0 \text{ and go to Step 1.}$$

b. If the difference in absolute value is greater than ΔK, go to Step 3.

c. If this is the first time the difference in absolute value is less than ΔK; set $K^* = M(C^*,\underline{S})$, $\underline{S}^* = \underline{S}$, $C_1 = C^*$ and go to Step 3. Otherwise, go to Step 3.

Step 3 - Set $\underline{S} = \underline{S} + \underline{\Delta S}$. If the new value of $\underline{S}$ is in the range of values of $\underline{S}$, go back to Step 1. Otherwise, if $K - M(C^*,\underline{S}) > \Delta K$ over all the range of values of S, set

$$\Delta C^* = \frac{\Delta C^*}{2} \ , \ C^* = C^* + \Delta C^*, \ \underline{S} = \underline{S}_0 \text{ and go back to Step 1; else}$$

the algorithm converges on the optimal values of C^* and $\underline{S}^*$.

For the particular example we are considering, when $K = 5.0$, we initiated the iterative search procedure as follows: The values of C^* were chosen over the range of values $[8.22, 8.27]$ in increment of 0.01; $S_1 = S_2$ values ranged over the closed interval $[3.8, 4.08]$ in increment of 0.02. The number of "steps" used in the numerical integration N_1 were selected to be 60 and 100. For the case $N_1 = 100$ we have

$C^* = 8.2625$ and $S_1^* = S_2^* = 4.02$

$M(C^*, \underline{S}^*) = 5.00694$ and

$$\frac{\partial M(C^*,\underline{S})}{\partial S_i} \ \bigg|_{\underline{S} = \underline{S}^*} = 0 \text{ for } (i = 1,2)$$

which compare favorably with the results obtained by Sivazlian [13] when following different optimal policies.

Graphically, the computed values (Appendix 5.A) of the function $M(C^*,\underline{S})$ are plotted in Figs. (5.4) and (5.5) for the cases when $N_1 = 60$ and 100 respectively. As an experimental observation, it may be noted that as the number of "steps" used in carrying out the numerical integration increases the curves become smoother.

The accuracy of the computed results is subject to two sources of error, truncation and roundoff. In computing the modified Bessel function $I_0(\underline{x})$, only the first ten terms of the convergent series were summed. However it was found experimentally that the roundoff error is the factor of significance and it is not easy to examine it analytically, since it is a function of the computer used and the size of the "steps" used in carrying out the numerical integration.

$N_1 = 60$

$M(C^*, \underline{S})$

$C^* = 8.26$

$C^* = 8.25$

$C^* = 8.24$

$C^* = 8.23$

$C^* = 8.22$

5.0

4.9

4.8

3.76   3.78

S_1

3.76
3.78

S_2

Fig. (5.4)

$N_1 = 100$

$M(C^*, \underline{S})$

$\leftarrow C^* = 8.27$
$\leftarrow C^* = 8.264$
$\leftarrow C^* = 8.20$
$\leftarrow C^* = 8.25$
$\leftarrow C^* = 8.24$
$\leftarrow C^* = 8.23$
$\leftarrow C^* = 8.22$

5.0

4.9

4.8

3.76

3.76
3.78

S_1

S_2
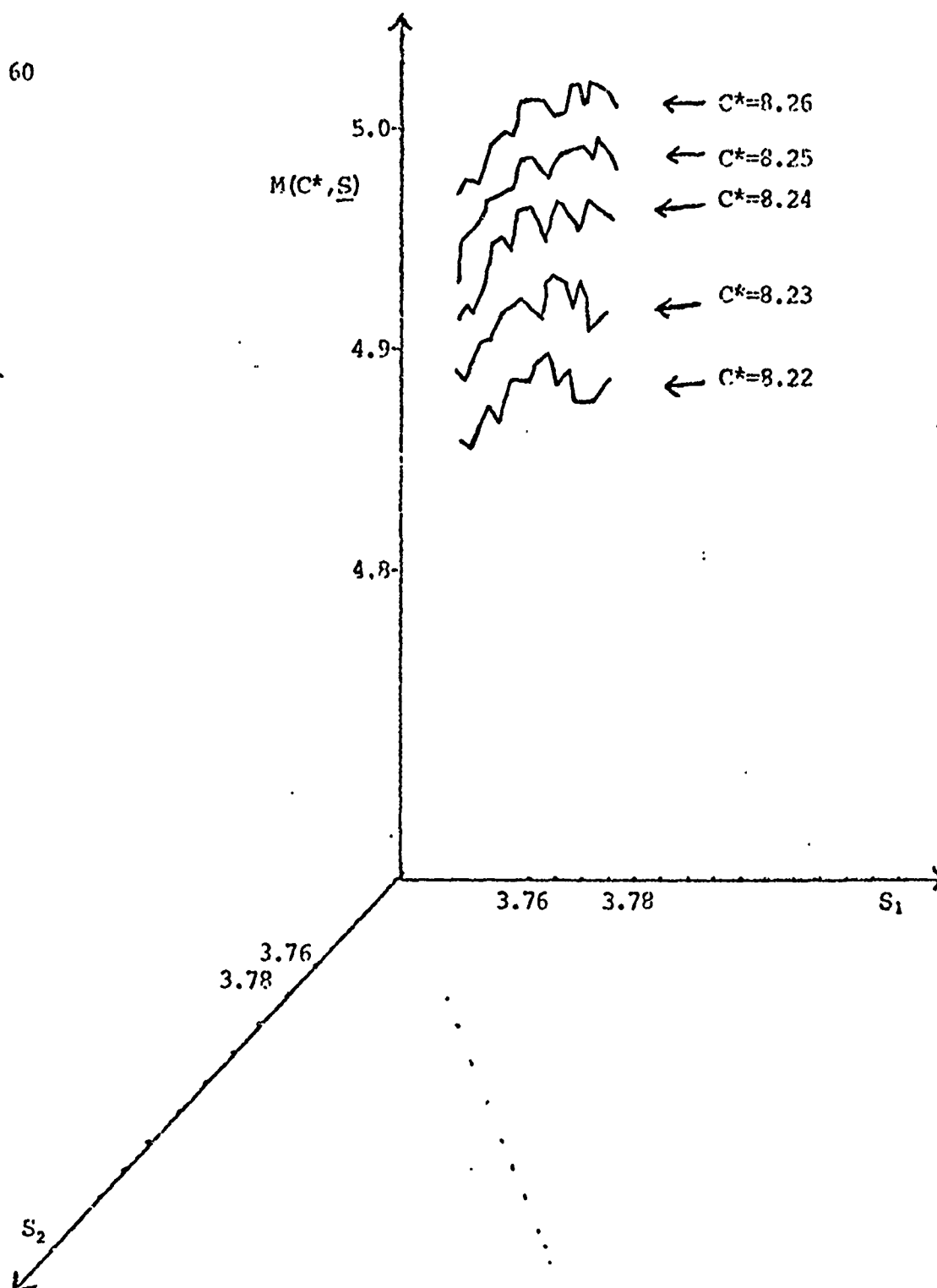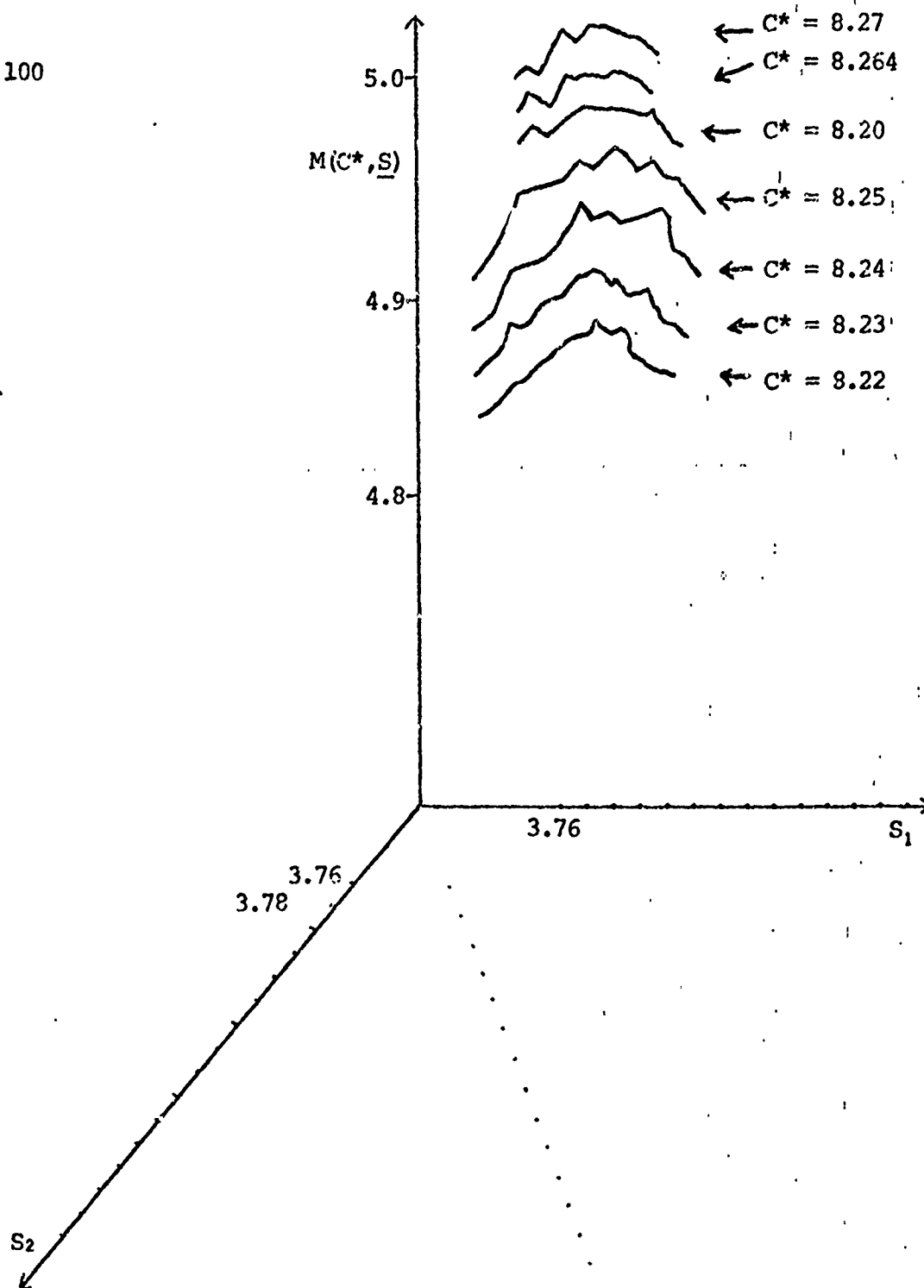
Fig. (5.5)

CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS

6.1 Conclusions

A stationary multi-commodity inventory problem with periodic review has been formulated and studied when a dyadic (σ,S) policy is used. At the beginning of each of a sequence of periods of time the stock level of each item is reviewed and a decision to order or not to order is made. The cost elements that affected the ordering deci- in each period are: the ordering costs, the holding cost and the shortage cost. The ordering costs are assumed to be composed of a purchasing cost which is proportional to the quantity of each item ordered and a single set up cost which is independent of the quantity ordered. The holding and the shortage costs are assumed to be charged on the basis of the stock levels at the end of the period. Demand, for the items, in each period of time is described by a continuous random vector, with a joint density function, independently distributed from period to period. Immediate delivery of orders and complete backlogging of all unfilled demands are assumed.

The inventory system just described was treated by many researchers under different optimal (σ,S) policies. This research, as compared with what has been done in this area, is more general in the sense no assumptions were made about the configuration of the ordering region and no specific joint density functions were considered.

The m-dimensional convolution operation which was introduced
in Chapter II and used in the study is a generalized concept of the
ordinary convolution operation. For the case when Γ, the hypersurface
that subdivides the ordering region and the not-ordering region, is
admissible, the g-convolution of any function reduces to the ordinary
convolution and all the g-convolution operation properties with the
Commutative Law hold. The g-convolution properties, in particular,
were used in the solution of an m-dimensional integral equation of
the renewal type, which is similar in form to the equation solved by
other researchers in this area but with no restrictions on the con-
figuration of Γ.

The analysis used in deriving the analytical expression for the
stationary level expected cost per period is similar to Sivazlian's [11]
approach for the one commodity problem when operating under a station-
ary policy of the (s,S) type. The asymptotic results for the problem
were deduced by appealing to the results of Yosida and Kakutan [22] in
the theory of linear operations.

At optimality it was shown that the set Γ^* that subdivides the
policy space, Ω, is given by

$$\Gamma^* = \{\underline{x} | \underline{x} \in \Omega; \; L(\underline{x}) - C^* = 0\}$$

where C^* is the minimum value of the stationary total expected cost
per period excluding the variable cost, and $L(\underline{x})$ is the conditional

expected holding and shortage cost function. By using this result, the minimization problem was reduced to finding the decision variables S_i^* ($i = 1,2,\ldots,m$) and C^* that minimize the stationary total expected cost per period. $\underline{S}^*$ and C^* were given as the real positive solutions to the set of simultaneous equations

$$M(C^*,\underline{S}) = K \qquad (6.1)$$

$$\left.\frac{\partial M(C^*,\underline{x})}{\partial x_i}\right|_{\underline{x}=\underline{S}^*} = 0 \qquad (i = 1,2,\ldots,m) \qquad (6.2)$$

which are similar in form to the set of equations used by Sivazlian [11] in determining the optimal values s^* and $Q^* = S^* - s^*$ for the one commodity problem. $M(C^*,\underline{x})$ satisfies the integral equation

$$M(C^*,\underline{x}) = C^* - L(\underline{x}) + \int_{R(\underline{x},C^*)} M(C^*,\underline{x-t})\ \phi(\underline{t})\ \underline{dt} \qquad (6.3)$$

in which ϕ is the joint density function of the demand. For the case of a two-commodity problem where the demand for the items obeyed the exponential distribution and the holding and shortage costs were linear, it was feasible to convert the integral equation (6.3) into a hyperbolic partial differential equation of the second order with boundary conditions. Analytical solution of the boundary value problem was then determined and used in (6.1) and (6.2) to determine S_1^*, S_2^* and C^*.

6.2 Extension and Recommendations

Several interesting directions for future research and development are suggested as a result of this research.

The computational aspects of the problem need further investigation. The necessary and sufficient conditions, established in this study, for the existence of a relative minima are valid for any continuous density function and any conditional expected holding and shortage cost function which is twice differentiable. In the case of a two-commodity problem, where the demand for the items obeys the exponential distribution and the holding and shortage costs are linear, it has been feasible to convert the integral equation into a partial differential equation of the second order with boundary conditions. However, what about the case when the demands for the two items are given by, say a gamma distribution? Is it then feasible to convert the integral equation into a partial differential equation? And if so, what computational procedure must be followed to solve for the policy parameters? Another question also arises, how to solve for the optimal policy parameters for the two-commodity problem, when the demand for the items are exponentially distributed and the holding and shortage costs are not linear but, say, quadratic.

As we have seen the determination of the optimum policy parameters is a complex task even in the case of a simple demand density function.

As an alternative to approximating these parameters by numerical means, the approach used by Roberts [3] could be followed to give analytical formulas for determining the asymptotic reordering region for the case when the holding and shortage costs are linear. In order to derive the asymptotic expressions, for the set of equations that is used to determine the policy parameters, an approximation for the renewal function $\int_{R(\underline{S},C)} \psi(\underline{t}) \, d\underline{t}$ and similar extensions of the type discussed in Smith [14] are required. The work of Bickel and Yahav [21] and Farrel [4] in investigating the asymptotic behavior of the renewal function in two and higher dimensions must be noted. However, the challenging problem is to find an analogous theorem to Smith's theorem involving the renewal density function in two and higher dimensions.

As noted in Chapter I, under the adopted (σ,S) policy, the sequence of stock levels at the beginning of each period forms a discrete Markov process. Greenberg's [5] approach could be followed to determine the transient distributions of the stock levels prior to ordering. For the stationary distributions Karlin [2] approach could be followed. The results of this research, especially the properties of the g-convolution operation, are hoped to be of great use in determining the transient and stationary distributions of the stock levels prior to making ordering decision.

This study treated "the Patient Customer Case" where all unfilled

demands were backlogged. For the lost case, the analysis can proceed along the same procedures used in Chapters III and IV.

GLOSSARY OF NOTATION

m the number of commodities considered

E^m m-dimensional $(m \geq 1)$ Euclidian-space

$\underline{D}$ a random vector representing the demand

D_i a random variable representing the demand of commodity i

Φ the probability distribution of D

Φ_i the probability distribution of D_i

ϕ a density function of D

ϕ_i a density function of D_i

$\underline{\mu}$ the mean vector o. D

$\Phi^{(n)}$ the n-fold ordinary convolution of Φ

$\Phi_i^{(n)}$ the n-fold ordinary convolution of Φ_i

$\phi^{(n)}$ the n-fold ordinary convolution of ϕ with itself

$\phi_{(n)}$ the n-fold generalized convolution of ϕ with itself

ψ $\psi(\underline{t}) = \sum\limits_{n=1}^{\infty} \phi_{(n)}(t)$, the renewal density function

σ the ordering region

σ^c the not-ordering region

Γ the set of points that separates σ and σ^c

$\underline{S}$ the point up to which we order

S_i the ith component of the vector $\underline{S}$

K the set up cost

c_i the unit purchase cost of product i

h_i the holding cost per unit of product i on hand at the end of a period

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p_i the shortage cost per unit of commodity i on hand at the end of a period

$L(\underline{x})$ the conditional expected shortage and holding cost measured at the end of a period

$g(\sigma,\underline{S})$ the stationary total expected cost per period

$I_n(x)$ the modified Bessel function of order n

*   starred symbols denote conditions at optimality

$C*$ the value of the total expected cost per period excluding the variable ordering cost

APPENDIX 5A

COMPUTED OUTPUT RESULTS

AND PROGRAM LIST

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```
┌─────────────────────────────────────────────────┐
│ - Choose initial values for C* and ΔC*          │
│ - Choose a range of values for S,               │
│   S∈[S₀,S⁰], so that for a given value          │
│   of C*, M(C*,S) achieves a maximum in          │
│   that range.                                   │
│ - Choose an initial value for ΔS                │
│ - Choose ΔK as an upper limit on K-M(C*,S)      │
└─────────────────────────────────────────────────┘
```

$$\text{Compute } M(C^*,\underline{S})$$

$$\text{Compute } \Delta M = M(C^*,\underline{S}) - K$$

Decision: $\Delta M > \Delta K$

— YES → Set $\Delta C^* = \Delta C^*/2$, $C^* = C^* - \Delta C^*$, $\underline{S} = S_0$

— NO → Decision: $|\Delta M| > \Delta K$

$|\Delta M| > \Delta K$ — YES → Set $\underline{S} = \underline{S} + \Delta S$

$|\Delta M| > \Delta K$ — NO → Decision: This is the first time $|\Delta M| \leq \Delta K$

This is the first time — YES → (goes down)

This is the first time — NO → Set $K^* = M(C^*,\underline{S})$, $S^* = \underline{S}$, $\overline{C}_1 = \overline{C}^*$

Decision: $\underline{S} > S^0$

$\underline{S} > S^0$ — NO → (loop back to Compute)

$\underline{S} > S^0$ — YES → Decision: For all $\underline{S} \in [S_0, S^0]$, $K - M(\overline{C}^*, \underline{S}) > \Delta K$

For all — YES → Set $\Delta C^* = \Delta C^*/2$, $C^* = \overline{C^* + \Delta C^*}$, $\underline{S} = \underline{S}_0$

For all — NO → STOP → The algorithm converges to $\underline{S}^*$ and $C^* = C_1$

```
C     SOLVING NUMERICALLY FOR THE OPTIMAL VALUES OF S1, S2, AND C
C     THE EXPECTED COST PER PERIOD
C     THE OPTIMAL VALUES SATISFIES M(C,S1,S2)=5.0
C     THE INTEGRATION OVER THE REGION W(C,S1,S2) IS DONE NUMERICALLY.
C     ALSO THE LINE INTEGRAL OVER AB IS COMPUTED NUMERICALLY
C     THE MODIFIED BESSEL FUNCTION IS COMPUTED BY SUMMING ONLY THE
C     FIRST TEN TERMS OF THE CONVERGENT SERIES, THUS ENCOUNTIRING
C     TRUNCATION ERROR.
C     THE ITERATIVE PROCEDURE FOR A GIVEN VALUE OF C COMPUTES
C     M(C,S1,S2) OVER A RANGE OF VALUES OF S1 AND S2. THIS PROCEDURE
C     IS CARRIED OVER A RANGE OF VALUES FOR C.
C     NOTE O M(C,S1,S2) IS A STRICTLY CONCAVE FUNCTION OF S1 AND S2.
      DIMENSION TR(30),T(30)
      DOUBLE PRECISION TOP,BOT,H1,H3,H4,Z,Z1,S,S1,S2,S3,VAR,C,P1,P2,
     1ANSW1,ANSW2,ANSW3,TR,T,X1,X2,Y1,Y3,Y4
      COMMON T
      N=100
      M=01
      KVAL=5
      K=10
      S1=3.90
      S2=3.90
      C=8.264
      C0=C
      H3=0.004
      H4=0.02
      TR(1)=1.
      T(1)=1.
      DO 70 IT=2,12
      TR(IT)=TR(IT-1)*IT
   70 T(IT)=TR(IT)**2
      TOP=1.071D9288
      BOT=1.687D27248
      ITR=0
```

```
26  X2=C+2.
    ITR=ITR+1
    IND=0
    INDX=0
    B1=S1
    B2=S2
    DO 15 JT=1,K
    Y4=0.
    Z1=0.
    B1=B1+H4
    B2=B2+H4
    ZI=BOT+((C-B1-B2)*TOP)
    VAR=B2-1.8
    H1=(B2-VAR)/N
    S=0.
    Y1=B2+H1
    DO 1 I=1,N
    Y1=Y1-H1
    IF(Y1 .EQ. VAR) GO TO 7
    X1=VAR+H1
    DO 1 J=1,N
    X1=X1-H1
    P1=0.
    P1=(B1-X1)*(B2-Y1)
    P2=2.*DSQRT(P1)
    ANSW1=0.
    CALL F1(P2,ANSW1)
    ANSW2=DEXP(X1+Y1-B1-B2)
    IF((X1+Y1+21.*DEXP(-X1)+21.*DEXP(-Y1)) .GT. X2) GO TO 5
    ANSW3=(C-X1-Y1)*ANSW1*ANSW2
    S=S+(ANSW3*H1*H1)
    GO TO 1
5   J=N
    ANSW3=(21.*DEXP(-X1)-1.)*ANSW1*ANSW2
```

```
      Y4=Y4+(ANSW3*H1)
    1 CONTINUE
    7 Z=0.
      Y1=VAR+H1
      DO 2 I=1,N
      Y1=Y1-H1
      Y3=VAR
      IF((Y3+Y1+21.*DEXP(-Y3)+21.*DEXP(-Y1)) .LE. X2) GO TO 8
      Y3=VAR+H1
      IF((Y3+Y1+21.*DEXP(-Y3)+21.*DEXP(-Y1)) .GT. X2) GO TO 11
    8 X1=Y3+H1
      DO 2 J=1,N
      X1=X1-H1
      P1=0.
      P1=(B1-X1)*(B2-Y1)
      P2=2.*DSQRT(P1)
      ANSW1=0.
      CALL F1(P2,ANSW1)
      ANSW2=DEXP(X1+Y1-B1-B2)
      IF((X1+Y1+21.*DEXP(-X1)+21.*DEXP(-Y1)) .GT. X2) GO TO 9
      ANSW3=(C-X1-Y1)*ANSW1*ANSW2
      Z=Z+(ANSW3*H1*H1)
      GO TO 2
    9 J=N
      ANSW3=(21.*DEXP(-X1)-1.)*ANSW1*ANSW2
      Y4=Y4+(ANSW3*H1)
    2 CONTINUE
   11 S3=(Z+Z1)+(2.*S)
      X1=VAR-H1
      DO 4 I=1,N
      X1=X1+H1
      IF(X1 .EQ. B1) GO TO 21
      Y1=VAR+H1
      DO 4 J=1,N
```

```
      Y1=Y1-H1
      IF((X1+Y1+21.*DEXP(-X1)+21.*DEXP(-Y1)) .GT. X2) GO TO 22
      GO TO 4
 22   J=N
      P1=0.
      P1=(B1-X1)*(B2-Y1)
      P2=2.*DSQRT(P1)
      ANSW1=0.
      CALL F1(P2,ANSW1)
      ANSW2=DEXP(X1+Y1-B1-B2)
      ANSW3=(21.*DEXP(-X1)-1.)*ANSW1*ANSW2
      Y4=Y4+(ANSW3*H1)
 4    CONTINUE
 21   S3=S3+Y4
      WRITE(6,40) ITR
      WRITE(6,10) S3,B1,C
      IF(DABS(S3-KVAL) .LE. 0.001) GO TO 31
      IF (S3 GT. KVAL) GO TO 32
      IND=1
      GO TO 15
 32   H3=H3/2.
      C=C-H3
      GO TO 26
 31   IF (INDX .EQ.1) GO TO 33
      INDX=1
 34   SA3=B1
      SA3=S3
      GO TO 15
 33   IF (SA3 .GE. S3) GO TO 15
      GO TO 34
 15   CONTINUE
      IF (INDX .EQ. 1) GO TO 35
      IF (IND .EQ. 0.) GO TO 38
      H3=H3/2.
```

```
      C=C+H3
      GO TO 26
35    WRITE(6,18) N
      WRITE(6,41)
      WRITE(6,39) CO
      WRITE(6,40) ITR
      WRITE (6,19)
      WRITE(6,10) SA3,SAB,C
38    STOP
39    FORMAT(1X,5HCO = E20.14)
41    FORMAT(1X,25HOPTIMUM VALUES OF C AND S)
40    FORMAT(1X,15HITERATION NO. = I2/)
10    FORMAT(1X,E20.14,1X,E20.14,1X,E20.14)
19    FORMAT(9X,43HC                              S1=S2          C/)
18    FORMAT(1H1,27H NUMBER OF SUBDIVISIONS N = I3)
      END
      SUBROUTINE F1(P2,ANSW1)
      COMMON T
      DIMENSION T(30)
      DOUBLE PRECISION P2,ANSW1,I,ALK
      ALK=0.
      ANSW1=1.
      DO 72 IJ=1,10
      ALK=((P2/2.)**(2*IJ))
72    ANSW1=ANSW1+(ALK/T(IJ))
      RETURN
      END
```

NUMBER OF SUBDIVISIONS N1 = 100
OPTIMUM VALUES OF C AND S
CO = 0.826399993896480 01
ITERATION NO. = 4

M(S,C)	S1=S2	C
0.500069499101990 01	0.401999664306640 01	0.826249994011600 01

ITERATION NO. = 1

0.49929868932991D 01   0.39199991226196E 01   0.82639999389648D 01
ITERATION NO. = 1

0.50025237931040D 01   0.39399986267090E 01   0.82639999389648D 01
ITERATION NO. = 2

0.49861731530005D 01   0.39199991226196E 01   0.82619999404997D 01
ITERATION NO. = 2

0.49963454724456D 01   0.39399986267090E 01   0.82619999404997D 01
ITERATION NO. = 2

0.49964193409612D 01   0.39599981307983E 01   0.82619999404997D 01
ITERATION NO. = 2

0.49976847270872D 01   0.39799976348877E 01   0.82619999404997D 01
ITERATION NC. = 2

0.49976411227791D 01   0.39999971389771E 01   0.82619999404997D 01
ITERATION NO. = 2

0.49970155160891D 01   0.40199966430664E 01   0.82619999404997D 01
ITERATION NO. = 2

0.49973585190972D 01   0.40399966430664E 01   0.82619999404997D 01
ITERATION NO. = 2

0.49926406941392D 01   0.40599956512451E 01   0.82619999404997D 01
ITERATION NO. = 2

0.49896597533353D 01   0.40799951553345E 01   0.82619999404997D 01
ITERATION NO. = 2

0.49809515723675D 01   0.40999946594238E 01   0.82619999404997D 01
ITERATION NO. = 3

0.49887571286414D 01   0.39199991226196E 01   0.82629999397323D 01
ITERATION NO. = 3

0.49887571286414D 01   0.39299991226196E 01   0.82619999404997D 01
ITERATION NC. = 3

0.49989796335317D 01   0.39399986267090E 01   0.82629999397323D 01
ITERATION NO. = 3

0.49984623388593D 01   0.39599981307983E 01   0.82629999397323D 01
ITERATION NO. = 3

0.50009154703708D 01   0.39799976348877E 01   0.82629999397323D 01

ITERATION NO. = 3

0.50018918310495D 01   0.39999971389771E 01   0.82629999397323D 01
ITERATION NO. = 3

0.50014878186614D 01   0.40199966430664E 01   0.82629999397323D 01
ITERATION NO. = 4

0.49880058218203D 01   0.39199991226196E 01   0.82624999401160D 01
ITERATION NO. = 4

0.49976382469257D 01   0.39399986267090E 01   0.82624999401160D 01
ITERATION NO. = 4

0.49976940840069D 01   0.39599981307983E 01   0.82624999401160D 01
ITERATION NO. = 4

0.50000390504699D 01   0.39799976348877E 01   0.82624999401160D 01
ITERATION NO. = 4

0.49990457161623D 01   0.39999971389771E 01   0.82624999401160D 01
ITERATION NO. = 4

0.50006949910199D 01   0.40199966430664E 01   0.82624999401160D 01
ITERATION NO. = 4

0.49993371212808D 01   0.40399961471558E 01   0.82624999401160D 01
ITERATION NO. = 4

0.49933495432785D 01   0.40599956512451E 01   0.82624999401160D 01
ITERATION NO. = 4

0.49908453297377D 01   0.40799951553345E 01   0.82624999401160D 01
ITERATION NO. = 4

0.49828455150016D 01   0.40999946594238E 01   0.82624999401160D 01

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